Math 23, Spring 2007
Lecture 8

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Outline

Material from last class

- Existence and uniqueness for linear second order equations
- Wronskian: solving initial value problems

The Wronskian and linear dependence

Definition

Two functions are said to be linearly dependent if we can find a linear combination which is identically zero. If no such linear combination exists, the functions are said to be linearly independent.

Example

$$y_1(t) = \cos^2(t), \; y_2(t) = 1 + \cos(2t)$$

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The Wronskian and linear dependence

Theorem

If f and g are differentiable functions on an open interval I and if $W(f, g, t_0) \neq 0$ for some $t_0 \in I$, then f and g are linearly independent on I. Moreover, if f and g are linearly dependent on I, then $W(f, g, t) = 0$ for all $t \in I$.

Q: How can we interpret this theorem in terms of our ability to solve an initial value problem?

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Abel's Theorem

Theorem

If y_1 and y_2 are solutions to

$$L[y] = y'' + p(t)y' + q(t)y = 0$$

where p and q are continuous on an open interval I then

$$W(y_1, y_2, t) = c\exp\left(-\int p(t)\,dt\right)$$

where c is a certain constant that depends on y_1 and y_2 but not t. Further, W is either identically zero on I (if $c = 0$) or is never zero.

Linear homogeneous second order constant coefficient equations

Complex Roots

When solving

$$ay'' + by' + cy = 0$$

we focused first on the case when there are two real roots, leaving two cases

1. $r_1 = r_2$ (a repeated root)
2. $b^2 - 4ac < 0$ (complex roots)

Linear homogeneous second order constant coefficient equations

Complex Roots

We will focus first on the second case:

$$r = \frac{-b}{2a} \pm i\frac{\sqrt{4ac - b^2}}{2a} = \alpha \pm i\beta$$

Such roots are called *complex conjugates* and, formally, the solutions to the ODE are given by

$$y_1(t) = e^{(\alpha+i\beta)t}, \; y_2(t) = e^{(\alpha-i\beta)t}$$

Q: How do we interpret these functions?

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Linear homogeneous second order constant coefficient equations

Euler's equation

We use Euler's equation:

$$e^{ix} = \cos(x) + i\sin(x)$$

Idea of the proof:

$$e^{ix} = \sum_{n=0}^{\infty} \frac{(ix)^n}{n!} = \sum_{n=0}^{\infty} (-1)^n \frac{x^{2n}}{(2n)!} + i \sum_{n=0}^{\infty} (-1)^{n-1} \frac{x^{2n-1}}{(2n-1)!}$$
$$= \cos(x) + i\sin(x)$$

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Linear homogeneous second order constant coefficient equations

Complex roots

Using this formula, we have

$$y_1 = e^{(\alpha+i\beta)t} = e^{\alpha t}(\cos(\beta t) + i\sin(\beta t))$$

$$y_2 = e^{(\alpha-i\beta)t} = e^{\alpha t}(\cos(\beta t) - i\sin(\beta t))$$

Notice that

$$\tilde{y}_1 = \frac{1}{2}(y_1 + y_2) = e^{\alpha t}(\cos(\beta t))$$

$$\tilde{y}_2 = \frac{1}{2i}(y_1 + y_2) = e^{\alpha t}(\sin(\beta t))$$

Q: Do these form a fundamental set of solutions?

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Q: Do these form a fundamental set of solutions?

1. Given two complex roots, $\alpha \pm i\beta$ of the characteristic equation of an ODE, show that the resulting solutions y_1 and y_2 are linearly independent for all values of t.
2. What does Abel's theorem tell us about the first question?
3. Solve

$$y'' + 4y = 0$$

subject to the initial conditions:

3.1 $y(0) = 0, y'(0) = 1$
3.2 $y(0) = 1, y'(0) = 1$

Work for next class

- Reading: 3.4,2.5
- Homework 3 is due monday 4/16