Math 23, Spring 2007
Lecture 5

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Outline

Last class

Today's material

- Modeling
- Population model: I
- Population model: II
- Population model: III
- Population model: III

Next class

Material from last class

- Numerical methods
- Euler's approximation method
- Using matlab to implement

Modeling

We have already briefly discussed modeling of “real world” situations via ODEs. Today, we will work on understanding the various aspects of modeling by examining a single model. Setps:

1. Pick the most relevant features of the system you wish to model
2. Define an equation which reflects those features
3. Solve the equation for a model solution
4. Compare solution to known data
5. If the error is not acceptable, refine the model

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Population model

Relevant features

Focus on a population model. How does a population grow over time. Our initial observation:

A population grows at a rate proportional to the population itself

Population model

Model ODE and solution

Using this idea we wrote down the model ODE

$$y' = ry$$

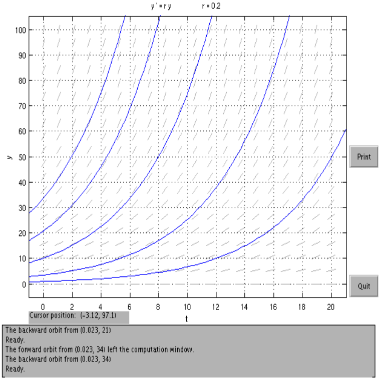
which is separable and has the solution

$$y(t) = e^{rt}$$

Population model

Analysis of the model

As we noted in class, this model is reasonable for small t, but is not appropriate for large t. In particular, $\lim_{t\to\infty} y(t) = \infty$ if $r > 0$.



Population model

Refine the model

If we would like the model to better reflect reality, we need to refine the model. To do so, we refine our defining feature of the model. Observations:

1. The proportionality of y and y' (i.e. r) need not be constant
2. For y small, r constant works well
3. For y large, this makes less sense. For example, a growing population will eventually exceed the resources available.
4. New model:

$$y' = h(y)y$$

where $h(y) \approx r$ when small and when $y >> 0$ $h(y)$ become negative.

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Population model

Refined model: the logistic model

We pick a simple $h(y)$ which satisfies these constraints: $h(y) = r(1 - y/K)$ where K is some constant. The graph of this function is a line passing through r at $y = 0$ and 0 at $y = K$. Our ODE becomes

$$y' = r\left(1 - \frac{y}{K}\right)y$$

Note: This is no longer a linear equation! However, it is separable.

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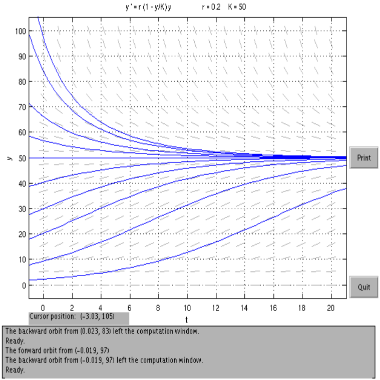
Population model

Sketching solutions

For a first order equation $y' = f(y)$

- Find equilibria: values of y where f is identically zero
- Sketch $f(y)$
- Use the graph to sketch solution curves by hand

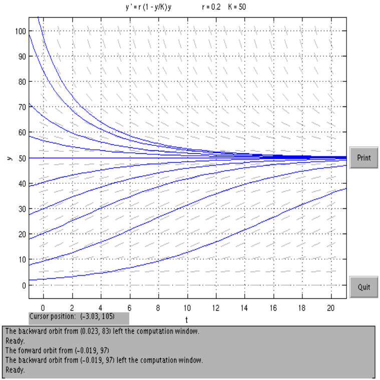


Population model

Observations

Questions:

1. How are the solutions here different from the linear model?
2. Are they more plausible?
3. Are there still inconsistencies with a real world situation?



Population model

Thresholds

In many populations, there is a threshold of viability, i.e. if the population falls below a certain threshold it will eventually become extinct. We can model this using either a perturbation of the linear model or the logistic model.

$$y' = -r\left(1 - \frac{y}{T}\right) y$$

or

$$y' = -r\left(1 - \frac{y}{T}\right)\left(1 - \frac{y}{K}\right) y$$

Class work: sketch $f(y)$ in each case and qualitatively describe the solutions

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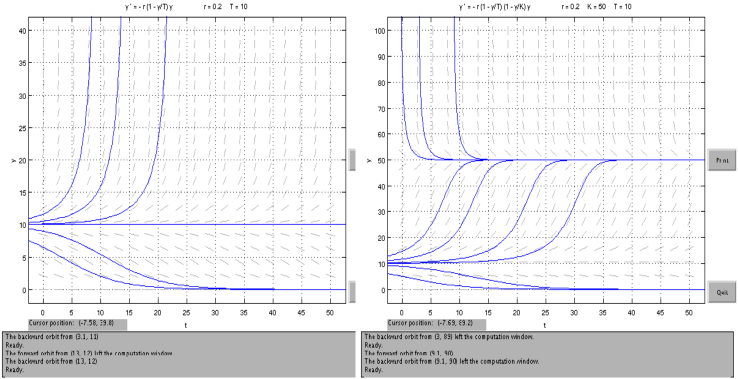
Population model

Thresholds

Results.



Work for next class

- Reading: 3.1
- Homework 2 is due monday