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Lecture 3

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Outline

Last class

Today's material

- Existence and Uniqueness
- Exact Equations

Group Work

Next class

Methods of solution

- Linear first order equations: integrating factors
- Separable Equations
- Fundamentally integration techniques

Uniqueness

Under what conditions can we determine that a solution is unique?

- Separable equations: one (combined) integration constant
- Linear first order equations: after simplification, one integration constant
- Specifying an initial value determines the constant
- Q: Does a first order initial value problem have a unique solution?

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A first Theorem

Theorem

If the function p and g are continouous on an open interval I : $\alpha < t < \beta$ *containing the point* $t = t_0$, *then there exists a unique function* $y = \phi(t)$ *that satisfies*

$$y' + p(t)y = g(t), \; y(t_0) = y_0$$

Example

Solve

$$y' + \frac{1}{2t}y = \sqrt{t}$$

subject to the following conditions:

1. $y(0) = 0$
2. $y(1) = 1$
3. $y(-1) = 0$

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An illustrative example

Solve

$$y' = y^{\frac{1}{3}}, \; y(0) = 0$$

A second theorem

Theorem

Let the function f and f_y be continuous on some rectangle $\alpha < t < \beta, \gamma < y < \delta$ containing the point (t_0, y_0). Then, in some interval $t_0 - h < t < t_0 + h$ contained in (α, β), there is a unique solution $y = \phi(t)$ of the initial value problem

$$y' = f(t, y), \ \ y(t_0) = y_0$$

Integrating factors

In the previous methods, we have used calculus rules to change the ODE into a form we can integrate (e.g. product rule, chain rule). We can use this philosophy more generally:

Example

Solve

$$2x + y^2 + 2xyy' = 0$$

by observing that

$$\frac{d}{dx}(x^2 + xy^2) = 2x + y^2 + 2xyy'$$

Integrating factors

In general, if our ODE is given by

$$M(x,y) + N(x,y)y' = 0$$

If we can find a ψ so that $\psi_x = M, \psi_y = N$ then $\psi(x,y) = C$ defines a solution to the equation. Such an equation is called *exact*.

Theorem

Let the function M, N, M_y, N_x be continuous on a rectangular region R. Then

$$M(x,y) + N(x,y)y' = 0$$

is an exact ODE on R if and only if

$$M_y = N_x$$

on all of R.

Further examples

Solve the following ODE

- $$(2x+3)+(2y-2)y' = 0$$
- $$(3x^2-2xy+2)+(6y^2-x^2+3)y' = 0$$

Work for next class

- Reading: 2.7
- Homework 2 is due 4/9
- Matlab intro tomorrow!