Math 23, Spring 2007

Lecture 24

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Material from last class

The heat equation

$$\alpha^2 u_{xx} = u_t$$

with conditions $u(x,0) = f(x), u(0,t) = u(L,t) = 0$.

1. Separate variables to get

$$\frac{X''}{X} = \frac{T'}{\alpha^2 T} = \lambda$$

   or

$$X'' - \lambda X = 0, \quad T' - \alpha^2 \lambda T = 0$$

2. Translation of boundary conditions

$$X(0) = 0 = X(L)$$

Eigenvalue problem

For the X equation, we have a two point boundary value problem

$$X'' - \lambda X = 0, \quad X(0) = 0 = X(L)$$

Solutions: $X_n(x) = \sin(n\pi x/L), \; n = 1, 2, 3, \ldots$ with eigenvalues $\lambda_n = -\frac{n^2\pi^2}{L^2}$.

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T equation

For these eigenvalues ($\lambda_n = -\frac{n^2\pi^2}{L^2}$), solve for T via

$$T' + \frac{\alpha^2 n^2 \pi^2}{L^2} T = 0$$

$$T_n(t) = e^{-\frac{\alpha^2 n^2 \pi^2}{L^2} t}$$

So, for each n we have a product solution to $\alpha^2 u_{xx} = u_t$ with $u(0,t) = u(L,t) = 0$:

$$u_n(x,t) = e^{-\frac{\alpha^2 n^2 \pi^2}{L^2} t} \sin(n\pi x/L)$$

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Superposition

Now, it is highly unlikely that

$$u_n(x,0) = \sin(n\pi x/L) = f(x)$$

so we are unable to impose the last condition on a single choice of solution. So, we use the principle of superposition

$$u(x,t) = \sum_{n=1}^{\infty} u_n(x,t)$$
$$= \sum_{n=1}^{\infty} c_n e^{-\frac{\alpha^2 n^2 \pi^2}{L^2} t} \sin(n\pi x/L)$$

This is a Fourier sine series when $t = 0$ so we need to represent f as such a series.

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Fourier sine series representation

Steps:

1. Make f odd an periodic with period $2L$
2. Compute coefficients

$$b_n = \frac{2}{L}\int_0^L f(x)\sin\left(\frac{n\pi x}{L}\right)\,dx$$

3. Then $c_n = b_n$

Example

$L = \pi$, $f(x) = (\pi - x)x^3$

http://www.math.cornell.edu/ bterrell/h1/heat1.html

Other types of boundary conditions

Insulated ends:

$$\alpha^2 u_{xx} = u_t, \;\; u_x(0,t) = u_x(L,t) = 0, u(x,0) = f(x)$$

Also called Neumann boundary conditions.
Two point boundary value problem:

$$X'' - \lambda X = 0, \;\; X'(0) = X'(L) = 0$$

and

$$T' - \alpha^2 \lambda T = 0$$

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Insulated ends

Solutions:

$$\lambda_n = n^2\pi^2/L, \quad X_n = \cos(n\pi x/L), \quad T_n = e^{-\frac{n^2\pi^2\alpha^2}{L^2}t}$$

So, $u_0(x,t) = 1$,

$$u_n(x,t) = e^{-\frac{n^2\pi^2\alpha^2}{L^2}t}\cos(n\pi x/L)$$

And

$$u(x,t) = 1 + \sum_{n=1}^{\infty} c_n e^{-\frac{n^2\pi^2\alpha^2}{L^2}t}\cos(n\pi x/L)$$

where

$$c_n = \frac{2}{L}\int_0^L f(x)\cos(n\pi x/L)\,dx$$

Work for next class

- Read 10.5,10.7,10.8
- Homework 9 assigned but is not due! These are practice problems for the final.