Math 23, Spring 2007

Lecture 22

Scott Pauls

Department of Mathematics
Dartmouth College

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Material from last class

- Two point boundary value problems
- Problems with existence and uniqueness
- Eigenvalues and Eigenfunctions

Laplace's equation

$$f_{xx} + f_{yy} = 0$$

Guess: $f(x, y) = X(x)Y(y)$

$$\frac{X''(x)}{X(x)} = \frac{Y''(y)}{Y(y)} = \lambda$$

Reorganizing this gives:

$$X'' - \lambda X = 0, \quad Y'' + \lambda Y = 0$$

For different values of λ, these equations have general solutions that look like either

$$C_1 \cos(\sqrt{|\lambda|}t) + C_2 \sin(\sqrt{|\lambda|}t)$$

or

$$C_1 \cosh(\sqrt{|\lambda|}t) + C_2 \sinh(\sqrt{|\lambda|}t)$$

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Series solutions

To form the most general solution, via the principle of superposition, we would take sums of these solutions over all eigenvalues λ yielding, for examples, infinite sums of sin and cos terms:

$$\frac{a_0}{2} + \sum_{n=1}^{\infty} a_n \cos\left(\frac{n\pi x}{L}\right) + b_n \sin\left(\frac{n\pi x}{L}\right)$$

Such a series is called a **Fourier series.** At points where it converges, it gives a function f. In this case, the series is called the Fourier series of f.

Fourier series

Some observations:

- Fourier series are periodic
- sin and cos functions are independent (also called **orthogonal**)) in the following sense:

$$\int_{-L}^{L} \cos\left(\frac{n\pi x}{L}\right) \sin\left(\frac{m\pi x}{L}\right)\ dx = 0$$

$$\int_{-L}^{L} \cos\left(\frac{n\pi x}{L}\right) \cos\left(\frac{m\pi x}{L}\right)\ dx = 0 \ \text{ if } m \neq n$$

If $m = n$, the integral equals L.

$$\int_{-L}^{L} \sin\left(\frac{n\pi x}{L}\right) \sin\left(\frac{m\pi x}{L}\right)\ dx = 0 \ \text{ if } m \neq n$$

If $m = n$, the integral equals L.

Euler-Fourier formulae

Assume for a moment that we have a convergent Fourier series:

$$f(x) = \frac{a_0}{2} + \sum_{n=1}^{\infty} a_n \cos\left(\frac{n\pi x}{L}\right) + b_n \sin\left(\frac{n\pi x}{L}\right)$$

What are the coefficients $\{a_n, b_n\}$?

$$a_n = \frac{1}{L}\int_{-L}^{L} f(x)\cos\left(\frac{n\pi x}{L}\right)\, dx$$

$$b_n = \frac{1}{L}\int_{-L}^{L} f(x)\sin\left(\frac{n\pi x}{L}\right)\, dx$$

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Examples

Find the Fourier series for

- $$f(x) = \begin{cases} 0, & -1 \leq x < 0 \\ 1, & 0 \leq x \leq 1 \end{cases}$$
- $$f(x) = \begin{cases} -x, & -1 \leq x < 0 \\ x, & 0 \leq x \leq 1 \end{cases}$$

Fourier Convergence Theorem

Theorem

Suppose f and f′ are piecewise continuous on the interval $-L \leq x < L$. Further suppose that f is defined outside this interval and is periodic with period $2L$. Then f has a Fourier series whose coefficients are given by our formulas above. The Fourier series converges to $f(x)$ at all points where f is continuous and to $(f(x+) + f(x-))/2$ at all points of discontinuity.

Work for next class

- Read 10.4
- Homework 8 is due Monday 5/21/07