Math 23, Spring 2007
Lecture 20

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Material from last class

- First order Linear Systems of equations with constant coefficients

$$\vec{x}' = A\vec{x}$$

- Cases:
  1. Two distinct real eigenvalues,

     $$\vec{x} = \xi_1 e^{r_1 t}, \vec{x} = \xi_2 e^{r_2 t}$$

     $r_1 \neq r_2$, both of same sign
     Description: equilibrium solution is a node, either asymptotically stable or unstable.
     $r_1 \neq r_2$, opposite signs
     Description: equilibrium solution is a saddle point
  2. Two equal eigenvalues, $r_1 = r_2$,
     Case 1: two independent eigenvectors

     $$\vec{x} = \xi_1 e^{r_1 t}, \vec{x} = \xi_2 e^{r_1 t}$$

     Description: proper node

Further cases

Case 2: $r_1 = r_2$, one eigenvector

$$\vec{x} = \xi_1 e^{r_1 t}, \vec{x} = \xi_1 t e^{r_1 t} + \eta e^{r_1 t}$$

Description: improper node

Complex eigenvalues: $r_1 = a + ib, r_2 = a - ib$

$$\vec{x} = \xi_1 e^{at} \cos(bt), \vec{x} = \xi_2 e^{at} \sin(bt)$$

Description: Spiral points ($a \neq 0$) and centers ($a = 0$)

Nonlinear systems

None of our methods currently apply for nonlinear systems but, just as we did for autonomous systems, we can use linear methods to help understand the nonlinear case.

- Critical points: $\vec{x}' = f(\vec{x})$. Find vectors so that $f(\vec{x}) = 0$.
- Assess stability:
  1. a critical point is stable if any solution that starts near the critical point stays near the critical point
  2. a critical point is asymptotically stable if it is stable and solutions tend to the critical point in the limit.
  3. unstable points are thos that are not stable.

Example: the oscillating pendulum

$$mL^2\frac{d^2\theta}{dt^2}+\gamma\frac{d\theta}{dt}+\omega^2\sin(\theta)$$

Converted to a system of first order equations:

$$x' = y$$

$$y' = -\omega^2\sin(x) - \gamma y$$

Find and classify critical points.

Example: the oscillating pendulum

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$$x' = y$$

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Find and classify critical points.

Finding trajectories

Example:

$$x' = 4 - 2y$$

$$y' = 12 - 3x^2$$

Critical points: $x = \pm 2, y = 2$ Rewrite:

$$\frac{dy}{dx} = \frac{12 - 3x^2}{4 - 2y}$$

Separate variables: $4y - y^2 + 12x + x^3 = C$

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Work for next class

- Read 10.1-10.2
- Homework 7 is due Monday 5/14/07