Math 23, Spring 2007

Lecture 2

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Outline

Last class

Today's material

- First Order Linear Equations
- Separable equations

Group Work

Next class

ODEs

- Introduced ordinary differential equations
- ODEs often arise from modeling physical situations
- Goal: develop methods for finding or approximating solutions
- First tool: direction fields

Initial value problems

Definition

An initial value problem is a set of two equations, an ODE and an equation specifying a value of the unknown function at zero.

Example

$$\frac{df}{dt} = f(t)$$
$$f(0) = 2$$

Basic Questions:

- Does a solution exist?
- How many solutions are there?

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A first order linear equation is an ODE of the form

$$\frac{dy}{dt} + p(t)y = g(t)$$

where $p(t), g(t)$ are fixed functions of t.

How can we solve this equation? Our only method is integration, but we can't simply integrate because of the dependence on y.

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Integrating factors

Main idea: make the left hand side of the equation,

$$\frac{dy}{dt} + p(t)y$$

look like the result of the product rule"

$$\frac{d}{dt}y(t)h(t) = \frac{dy}{dt}h(t) + \frac{dh}{dt}y$$

To do this, we multiply by a factor $\mu(t)$:

$$\mu(t)\frac{dy}{dt} + \mu(t)p(t)y$$

To make this work, we need to find the correct μ so that $\mu(t)p(t) = \mu'(t)$.

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If we can find such a μ, then our ODE can be written as

$$\frac{d}{dt}\mu(t)y(t) = \mu(t)g(t)$$

and, after integrating, we have

$$y(t) = \frac{\int \mu(t)g(t)\,dt}{\mu(t)}$$

Integrating factors

So, we are now left with solving the auxillary ODE

$$\mu'(t) = \mu(t)p(t)$$

We can solve this using integration:

$$\int \frac{\mu'(t)}{\mu(t)}\, dt = \int p(t)\, dt$$

$$\ln(\mu(t)) = \int p(t)\, dt$$

Simplifying,

$$\mu(t) = e^{\int p(t)\, dt}$$

First Order Linear Equations

Solutions via integrating factors

Given an initial value problem

$$\frac{dy}{dt} + p(t)y = g(t), \ \ y(0) = y_0$$

The solution is

$$y(t) = e^{-\int p(t)\,dt} \int e^{\int p(t)\,dt} g(t)\,dt$$

where the constant of integration is determined by requiring $y(0) = y_0$.

Separable equations

Another special class of ODEs:

$$M(x) + N(y)\frac{dy}{dx} = 0$$

where M, N are arbitrary functions of one variable. To solve this, we write this as

$$M(x)\, dx = -N(y)\, dy$$

Integrate both sides

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and solve for y (if possible).

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$$\frac{dy}{dx} + \sin(x)y^2 = 0, \; y(0) = 1$$

Further examples

Solve the following ODE

- $$\frac{dy}{dx} - \frac{x^2}{1+y^2} = 0$$
- $$\frac{dy}{dx} - x - 2xe^{2x} = 0$$
- $$\frac{dy}{dx} - y - 2xe^{2x} = 0$$

Work for next class

- Reading: 2.4-2.6
- Homework 1 is due 4/2