Math 23, Spring 2007
Lecture 18

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Material from last class

- First order Linear Systems of equations with constant coefficients

$$\vec{x}' = A\vec{x}$$

- Method of solution: guess $\vec{x} = \xi e^{rt}$ which relates solutions to the eigenvalues and eigenvectors of A

Equilibrium points

For

$$\vec{x} = A\vec{x}$$

we have any $\vec{x}$ with $A\vec{x} = 0$ as an equilibrium point. In particular, $\vec{x} = 0$ is always such a point.

Classification: let λ_1, λ_2 be the eigenvalues of A

1. $\lambda_1 < 0 < \lambda_2$, real: zero is a **saddle point** and is an unstable equilibrium point.
2. λ_1, λ_2 real, nonzero and of the same sign: zero is a **node** is asymptotically stable, if $\lambda_i < 0$, and asympotically unstable otherwise

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Equilibrium points

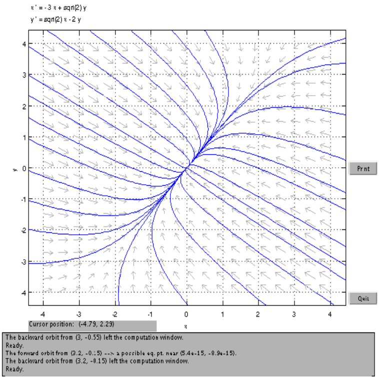

Figure: A saddle point

Equilibrium points

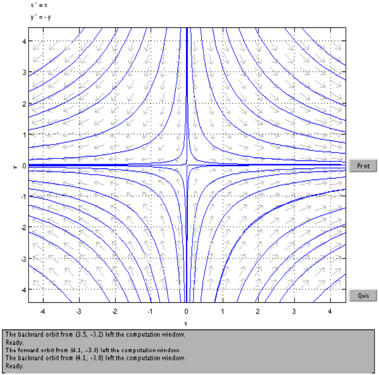

Figure: A node (stable)

Complex eigenvalues

As we have seen, we may have two complex conjugate eigenvalues:

$$\vec{x}' = \begin{pmatrix} 3 & -2 \\ 4 & -1 \end{pmatrix} \vec{x}$$

Eigenvalues: $1 \pm 2i$
Solution:

$$\vec{x} = \xi e^{(1 \pm 2i)t}$$

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Complex eigenvalues

Recall Euler's equation:

$$e^{it} = \cos(t) + i\sin(t)$$

Suppose we have eigenvalues $\alpha \pm i\beta$ and eigenvectors $a \pm ib$. Then

$$\vec{x} = (a+ib)e^{(\alpha+i\beta)t}$$
$$= e^{\alpha t}((a\cos(\beta t) - b\sin(\beta t)) + i(a\sin(\beta t) + b\cos(\beta t)))$$

and

$$(a-ib)e^{(\alpha-i\beta)t}$$
$$= e^{\alpha t}((a\cos(\beta t) + b\sin(\beta t)) + i(-a\sin(\beta t) - b\cos(\beta t)))$$

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Complex eigenvalues

Taking an appropriate linear combination yields two real functions

$$u(t) = e^{\alpha t}((a\cos(\beta t) - b\sin(\beta t))$$

$$v(t) = e^{\alpha t}((a\sin(\beta t) + b\cos(\beta t))$$

Caution: remember that *a* and *b* are vectors.

Example

$$\vec{x}' = \begin{pmatrix} 3 & -2 \\ 4 & -1 \end{pmatrix} \vec{x}$$

Classification of equilibrium points

1. $\lambda = a \pm ib$: zero is a **spiral point** and is asympototically stable if $a < 0$ and unstable otherwise.
2. $\lambda - a \pm ib$, $a = 0$: zero si called a **center** and is stable but not asymptotically stable.

Equilibrium points

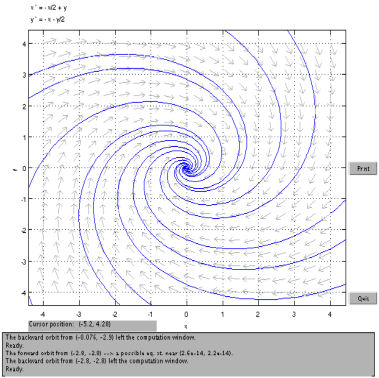

Figure: A spiral point: asympt. stable

Equilibrium points

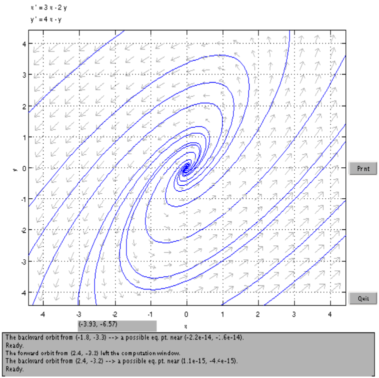

Figure: A sprial point:asympt. unstable

Equilibrium points

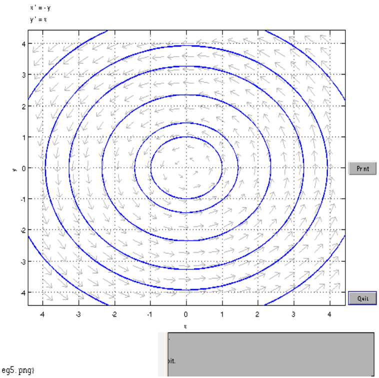

Figure: A center

Example

$$\vec{x}' = \begin{pmatrix} 1 & 0 & 0 \\ 2 & 1 & -2 \\ 3 & 2 & 1 \end{pmatrix} \vec{x}$$

Work for next class

- Read 7.7
- Homework 7 is due Monday 5/14/07