# Math 23, Spring 2007

## Lecture 17

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# Material from last class

- First order systems of equations
- Linear algebra review
- Eigenvalues and Eigenvectors

# Linear first order systems

$$
\begin{aligned}
x_1' &= p_{11}(t)x_1 + \cdots + p_{1n}(t)x_n + g_1(t) \\
x_2' &= p_{21}(t)x_1 + \cdots + p_{2n}(t)x_n + g_2(t) \\
&\vdots \\
x_1' &= p_{n1}(t)x_1 + \cdots + p_{nn}(t)x_n + g_n(t)
\end{aligned}
$$

Rewrite this in matrix form:

$$\vec{x}' = P(t)\vec{x} + \vec{g}(t)$$

The system is *homogeneous* if $\vec{g}$ is the zero vector. We will write a solution to this system as

$$\vec{x} = \vec{\phi}(t) = \begin{pmatrix} \phi_1(t) \\ \vdots \\ \phi_n(t) \end{pmatrix}$$

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# Linear first order systems

Principle of superposition

## Theorem

*If $p\vec{h}i(t)$ and $\vec{\psi}(t)$ are solutions to a homogeneous linear first order system, then any linear combination is also a solution.*

## Theorem

*A set of solutions to a first order linear system, $\{\vec{\phi}_1(t), \ldots, \vec{\phi}_n(t)\}$ are linearly independent at t if*

$$W(\vec{\phi}_1(t), \ldots, \vec{\phi}_n(t)) = det \begin{pmatrix} \phi_{11}(t) & \ldots & \phi_{1n}(t) \\ \vdots & \vdots & \vdots \\ \phi_{n1}(t) & \ldots & \phi_{nn}(t) \end{pmatrix} \neq 0$$

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# Linear homogeneous first order systems

## Theorem

*If $\{\vec{\phi}_1(t), \dots, \vec{\phi}_n(t)\}$ are linearly independent solutions to a linear first order homogeneous system for $\alpha < t < \beta$, then every solution to the system may be uniquely written as a linear combination of the $\vec{\phi}_i$.*

## Theorem

*If $\{\vec{\phi}_1(t), \dots, \vec{\phi}_n(t)\}$ are solutions to a linear first order homogeneous system on the interval $\alpha < t < \beta$ then $W(\vec{\phi}_1(t), \dots, \vec{\phi}_n(t))$ is either identically zero on this interval or it never vanishes.*

# Methods for finding solutions

## Phase planes and phase portraits

$$\vec{x}' = A\vec{x}$$

where $A$ is an $n \times n$ matrix.
Cases:
n=1: this is a single linear first order equation $x' = ax$.
Methods of solution: Direction fields, integrating factors

n=2: this is a pair of linear equations

$$x' = ax + by$$
$$y' = cx + dy$$

Draw phase plane/portrait. Use `pplane7.m`.

# Methods for finding solutions

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# Methods for finding solutions

$$\vec{x}' = A\vec{x}$$

Idea: mimic solutions for second order equations, guess

$$\vec{x} = \vec{\xi} e^{rt}$$

Plug into the equation:

$$r\vec{\xi} e^{rt} = A\vec{\xi} e^{rt}$$

Divide through by $e^{rt}$ and rewrite as

$$(A - rI)\vec{\xi} = 0$$

Conclusion: solutions of this form are determined by the eigenvalues and eigenvectors of $A$.

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# Example

$$\vec{x}' = \begin{pmatrix} 1 & 1 \\ 4 & 1 \end{pmatrix} \vec{x}$$

# Work for next class

- Read 7.6
- Homework 6 is due Monday 5/7/07