Math 23, Spring 2007
Lecture 13

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4/25/07

Last class

Today's material
Resonance
General second order linear equations
Series Solutions

Next class

Outline

Last class

Today's material

- Resonance
- General second order linear equations
- Series Solutions

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Material from last class

- Spring-mass systems with forcing

$$mu'' + \gamma u' + ku = F_0 \cos(\omega t)$$

- No damping: amplitude modulation
- Damping: resonance when ω_0 is close to ω

Pendulum

The pendulum is modeled by the ODE

$$\frac{d^2\theta}{dt^2} + \frac{g}{L}\sin(\theta) = 0$$

which we can reduce to a linear version (for small θ):

$$\frac{d^2\theta}{dt^2} + \frac{g}{L}\theta = 0$$

Solution: $r = \pm i\sqrt{g/L} = \pm i\omega$

$$y_c(t) = c_1\cos(\omega t) + c_2\sin(\omega t)$$

If we add forcing, $F_0\cos(\omega_0 t)$, we expect the largest effect when ω_0 is close to ω.

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Second order linear equations

So far, we have focused on the constant coefficient second order equations:

$$ay'' + by' + cy = g(t)$$

but we do not have any methods for more general linear equations:

$$y'' + p(t)y' + q(t)y = g(t)$$

or even more general equations

$$y'' = f(y, y', t)$$

Example: for the pendulum equation, we *approximated* the equation by a constant coefficient linear version by replacing $\sin(\theta)$ with θ.

Two basic ideas:

- Approximate general equations by linear ones
- Generate approximate solutions to linear equations which converge to exact solutions.

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Power series solutions

For a linear equation:

$$y'' + p(t)y' + q(t)y = g(t)$$

finding an exact solution is often too difficult. To create a general method of solution, we represent the solution as a power series

$$y(t) = \sum_{n=0}^{\infty} a_n (t - t_0)^n$$

Our goal is to

- Find the a_n
- Find the radius of convergence of the resulting power series.

Brief review of power series

A function $y(t)$ is said to be represented by a power series on the interval I if

$$y(t) = \sum_{n=0}^{\infty} a_n (t - t_0)^n$$

for some coefficients $\{a_n\}$ and all $t \in I$.

Finding power series:

- Taylor's formula

$$y(t) = y(t_0) + \sum_{n=1}^{\infty} \frac{y^{(n)}(t_0)}{n!} (t - t_0)^n$$

- Build series from known series via substitution, integration or differentiation

Radius of convergence:

- Ratio test

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Finding power series solutions

The basic idea is simple,

1. Substitute $y(t) = \sum_{n=0}^{\infty} a_n(t - t_0)^n$ into the ODE
$$y'' + p(t)y' + q(t)y = g(t)$$
2. Replace p, q, g with power series representations expanded about t_0
3. Expand and simplify
4. Solve for the a_n

Examples

- $y'' + y = 0$
- $y'' + ty = 0$

Work for next class

- Read: 5.1-5.3
- Homework 5 is due wednesday 5/1

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