Math 23, Spring 2007

Lecture 11

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Outline

Last class

Today's material

- Applications: mass-spring system
- Interpretation of solutions
  - No forcing: $F_e(t) = 0$
  - Forcing
- Examples

Next class

Material from last class

- Inhomogeneous equations
- Method of undetermined coefficients
- Variation of parameters

Mass spring system

Consider a mass on the end of a spring. It is pulled downwards by gravity. When in motion, the spring resists deformation, attempting to push the system back into equilibrium.

Modeling:

- Newton's law: $F = ma$
- Let $u(t)$ be the displacement of the mass from equilibrium
- Newton's law reads: $mu''(t) = F$ where F is the sum of the forces on the mass.
- Forces: gravity, resistence of spring, other forces

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Mass-spring system

Modeling forces

Gravity: $F_g = mg$ where g is the acceleration due to gravity

Resistence of spring: Hooke's law: the force due to the spring is proportional to the elongation (or compression) of the spring away from equilibrium. $F_s = -k(L+u)$ where L is the equilibrium length of the spring and k is a physical constant associated to the spring.

Damping: Damping works opposite to the motion of the force and is proportional to the velocity. $F_d = -\gamma u'$, γ is a physical constant.

External forces: F_e.

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Mass-spring systems

Some details: What is the equilibrium length L?
L is achieved when the forces balance (without external forces): $mg - k(L+0) = 0$ or $L = mg/k$.
Collect this information together:

$$\begin{aligned} mu''(t) &= F_g + F_s + F_d + F_e \\ &= mg - k(L + u(t)) - \gamma u'(t) + F_e(t) \\ &= -ku(t) - \gamma u'(t) + F_e(t) \end{aligned}$$

We may rewrite this as

$$mu'' + \gamma u' + ku = F_e(t)$$

Look familiar?

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Undamped free vibrations: $\gamma = 0$

$$mu'' + ku = 0$$

Solutions:

$$u(t) = A\cos(\omega_0 t) + B\sin(\omega_0(t))$$

$\omega_0^2 = k/m$.

$\omega + 0$ is called the *natrual frequency* of the vibration.
$\frac{2\pi}{\omega_0}$ is the period
$R = \sqrt{A^2 + B^2}$ is the amplitude of the vibration

Damping $\gamma \neq 0$

$$mu'' + \gamma u' + ku = 0$$

Cases:

1. $\gamma^2 - 4km \geq 0$, two real roots, general solution

$$u(t) = Ae^{r_1 t} + Be^{r_2 t}$$

if $r_1 \neq r_2$ and

$$u(t) = (A + Bt)e^{r_1 t}$$

if $r_1 = r_2$
Since $\gamma, k, m > 0$, $r_1, r_2 < 0$. These cases are called *overdamping* and *critical damping respectively*.

2. $\gamma^2 - 4kn < 0$, two complex roots, general solution

$$u(t) = e^{-\frac{\gamma t}{km}}(A\cos(\mu t) + B\sin(\mu t))$$

where $\mu = \frac{\sqrt{4km-\gamma^2}}{km}$. This case is called *underdamping*.

Forcing

If F_e is not zero, our model produces an inhomogeneous equation

$$mu'' + \gamma u' + ku = f(t)$$

This will introduce new and interesting behavior which we will investigate next class.

Example

Begin with no damping, no external force and unit mass. Assume that at equilibrium, $L = 2$.

1. What is the model ODE? what is its solution?
2. Now add damping, $\gamma = 6, 2\sqrt{5}, 2$. Same questions.
3. For each of these consider the initial conditions $u(0) = 0, u'(0) = 1$, $u(0) = 1, u'(0) = 0$. Qualitatively, what do the solutions look like?

Work for next class

- Reading:
- Homework 4 is due monday 4/23
- Midterm exam: Next tuesday. Covers through last class (section 3.7)
- See webpage for review materials