40 1. Consider the spring-mass system for a mass $m = 1$ described by the equation:

$$u'' + 5u' + 6u = 3e^{-t}$$

(a) Find the transient solution.

transient solution is solution to $u''+5u'+6u=0$

Char. eq.: $r^2+5r+6=0$

$(r+2)(r+3)=0$

$r=-2,-3$

$\boxed{u_c = c_1 e^{-2t} + c_2 e^{-3t}}$

(b) Find the steady state solution using the method variation of parameters.

Method 1

Let $u = \mu_1 e^{-2t} + \mu_2 e^{-3t}$

Set $\mu_1' e^{-2t} + \mu_2' e^{-3t} = 0$ $\quad \mu_1' = -\mu_2' e^{-t}$

$-2\mu_1' e^{-2t} - 3\mu_2' e^{-3t} = 3e^{-t}$

substitute:

$-2(-\mu_2' e^{-t})e^{-2t} - 3\mu_2' e^{-3t} = 3e^{-t}$

$-\mu_2' e^{-3t} = 3e^{-t}$

$\mu_2' = -3e^{2t}$, $\mu_1' = -(-3e^{2t})e^{-t} = 3e^{t}$

$\mu_2 = -\frac{3}{2}e^{2t} + c_2$, $\mu_1 = 3e^{t} + c_1$

So $u = c_1 e^{-2t} + c_2 e^{-3t} + 3e^{-t} - \frac{3}{2}e^{-t}$

steady state solution: $3e^{-t} - \frac{3}{2}e^{-t}$

$= \boxed{\frac{3}{2}e^{-t}}$

Method 2

Let $u = \mu_1 y_1 + \mu_2 y_2$ $\quad y_1 = e^{-2t}, y_2 = e^{-3t}, g = 3e^{-t}$

$\mu_1 = -\int \frac{y_2 \cdot g\,dt}{W(y_1,y_2)} + c_1$, $\mu_2 = \int \frac{y_1 \cdot g\,dt}{W(y_1,y_2)} + c_2$

$W(y_1,y_2) = \begin{vmatrix} e^{-2t} & e^{-3t} \\ -2e^{-2t} & -3e^{-3t} \end{vmatrix} = -3e^{-5t} + 2e^{-5t} = -e^{-5t}$

$\mu_1 = -\int \frac{e^{-3t} \cdot 3e^{-t}}{-e^{-5t}}dt + c_1 = -\int \frac{-3}{e^{-t}}dt + c_1 = \int 3e^{t}dt + c_1$

$= 3e^{t} + c_1$

$\mu_2 = \int \frac{e^{-2t} \cdot 3e^{-t}}{-e^{-5t}}dt + c_2 = \int \frac{-3}{e^{-2t}}dt + c_2 = \int -3e^{2t}dt + c_2$

$= -\frac{3}{2}e^{2t} + c_2$

$u = c_1 e^{-2t} + c_2 e^{-3t} + 3e^{-t} - \frac{3}{2}e^{-t}$

steady state solution: $3e^{-t} - \frac{3}{2}e^{-t}$

$= \boxed{\frac{3}{2}e^{-t}}$

(c) Suppose we want to change the damping so that the system is not critically damped or overdamped. If everything else remains the same what are the possible values of the damping constant?

critically damped means $\gamma^2 - 4km = 0$
overdamped means $\gamma^2 - 4km > 0$

So we want $\gamma^2 - 4km < 0$

$\gamma^2 < 4km$

$-\sqrt{4km} < \gamma < \sqrt{4km}$ $\qquad k = 6,\ m = 1$

So $-\sqrt{24} < \gamma < \sqrt{24}$

But damping constants can't be negative

so $\boxed{0 \leq \gamma < \sqrt{24}}$

or $\boxed{0 \leq \gamma < 2\sqrt{6}}$

25 2. Consider the spring-mass system from the previous problem.

(a) Write the equation we obtain if we change the damping constant and the external force to 0.

$$\boxed{u'' + 6u = 0}$$

(b) Using this new equation and the initial conditions $u(0) = -1$, $u'(0) = 3\sqrt{2}$ find the period, frequency, amplitude, and phase of the solution. Reduce your answers as much as possible.

Char. eq.: $r^2 + 6 = 0$

$r^2 = -6$

$r = \pm i\sqrt{6}$

$u = c_1 \cos(\sqrt{6}\,t) + c_2 \sin(\sqrt{6}\,t)$

$u' = -\sqrt{6}\,c_1 \sin(\sqrt{6}\,t) + \sqrt{6}\,c_2 \cos(\sqrt{6}\,t)$

plug in initial conditions

$-1 = c_1$

$3\sqrt{2} = \sqrt{6}\,c_2$, $c_2 = \frac{3\sqrt{2}}{\sqrt{6}} = \frac{3}{\sqrt{3}} = \sqrt{3}$

$u = -\cos(\sqrt{6}\,t) + \sqrt{3}\sin(\sqrt{6}\,t)$

for solution of form $u = A\cos(\omega t) + B\sin(\omega t)$

frequency $= \omega$, period $= \frac{2\pi}{\omega}$, amplitude $= \sqrt{A^2+B^2}$, phase is δ where $\tan\delta = \frac{B}{A}$

frequency: $\sqrt{6}$

period: $\frac{2\pi}{\sqrt{6}}$

amplitude: $\sqrt{(-1)^2 + (\sqrt{3})^2} = 2$

phase: $\tan\delta = \frac{\sqrt{3}}{-1}$ so $\delta = \frac{2\pi}{3}$

$\boxed{15}$ 3. What is a lower bound on the radius of convergence of the power series solution to $(x^2+1)(x+1)y'' + xy' - 3y = 0$ centered at $x = 1$?

In standard form this is

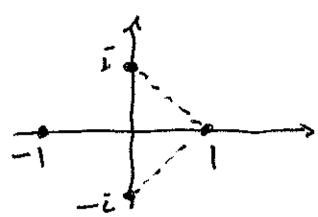
$$y'' + \frac{x}{(x^2+1)(x+1)} y' - \frac{3}{(x^2+1)(x+1)} y = 0$$

and the solutions are guaranteed to exist when the coefficient functions are analytic (in $\mathbb{C}$), i.e.

$$(x^2+1)(x+1) \neq 0$$

$$x \neq -1, \pm i$$

Complex plane:



The closest poles to the center $x=1$ is $x = \pm i$
which are at distance $\sqrt{1^2+1^2} = \sqrt{2}$
so a lower bound on the R.O.C. is $R = \sqrt{2}$.

$\boxed{30}$ 4. Let $y = \sum_{n=0}^{\infty} a_n x^n$ be a power series solution to the equation

$$y'' + x^2 y' + 2xy = 0$$

(a) Show that $a_2 = 0$ and that the coefficients satisfy $a_{n+3} = -\frac{a_n}{n+3}$ for all $n \geq 0$.

$$y' = \sum_{n=0}^{\infty} n a_n x^{n-1}$$

$$y'' = \sum_{n=0}^{\infty} n(n-1) a_n x^{n-2}$$

so $$\sum_{n=0}^{\infty} n(n-1) a_n x^{n-2} + x^2 \sum_{n=0}^{\infty} n a_n x^{n-1} + 2x \sum_{n=0}^{\infty} a_n x^n = 0$$

$$\sum_{n=-3}^{\infty} (n+3)(n+2) a_{n+3} x^{n+1} + \sum_{n=0}^{\infty} (n+2) a_n x^{n+1} = 0$$

$$2a_2 + \sum_{n=0}^{\infty} (n+3)(n+2) a_{n+3} x^{n+1} + \sum_{n=0}^{\infty} (n+2) a_n x^{n+1} = 0$$

$$2a_2 + \sum_{n=0}^{\infty} (n+2)\left[(n+3)a_{n+3} + a_n\right] x^{n+1} = 0$$

so by uniqueness of power series,

$a_2 = 0$

and $(n+3)a_{n+3} + a_n = 0$

so $a_{n+3} = \frac{-a_n}{n+3}$

(b) Use the recurrence relation to find a power series solution to the initial value problem

$$y'' + x^2y' + 2xy = 0, y(0) = 1, y'(0) = 0$$

Express your answer in summation notation.

$a_0 = 1,\ a_1 = 0$

$a_2 = 0$ so $a_{3n+2} = 0$ for all n

$a_1 = 0$ so $a_{3n+1} = 0$ for all n

$a_3 = \frac{-1}{3}$

$a_6 = -\frac{(-1/3)}{6} = \frac{1}{3\cdot 6}$

$a_9 = -\frac{1/3\cdot 6}{9} = \frac{1}{3\cdot 6\cdot 9}$

$a_{3n} = \frac{1}{3\cdot 6 \cdots (3n)} = \frac{(-)^n}{3^n n!}$

so $y = \sum_{n=0}^{\infty} \frac{(-1)^n}{3^n n!} x^{3n} = \sum_{n=0}^{\infty} \frac{(-x^3/3)^n}{n!} = e^{-x^3/3}$

(c) What is $y^{(101)}(0)$? Why?

$101 = 3\cdot 33+2$ so $a_{101} = 0$

$a_{101} = \frac{y^{(101)}(0)}{101!}$

so $y^{(101)}(0) = 0$.

$\boxed{20}$ 5. An Euler equation $L[y] = 0$ has complex solution $y = x^{2+3i}$. Show how to obtain a fundamental set of real solutions to the equation. *You do not have to justify that your answer is a fundamental set.*

$$x^{2+3i} = x^2 x^{3i}$$
$$= x^2 \left(e^{\ln(x)}\right)^{3i}$$
$$= x^2 \left(e^{i \cdot 3\ln(x)}\right)$$
$$= x^2 \left(\cos(3\ln(x)) + i \sin(3\ln(x))\right)$$

The real & imaginary parts are both solutions to a linear eqn. with real coefficients, so

$$x_1 = x^2 \cos(3\ln(x))$$
$$x_2 = x^2 \sin(3\ln(x))$$

are a fund. set of solutions.

$\boxed{20}$ 6. Find the general solution of $(x-2)^2y'' + 7(x-2)y' + 9y = 0$.

You should not have to use the method of power series to solve this.

Let $y = (x-2)^r$

$y' = r(x-2)^{r-1}$

$y'' = r(r-1)(x-2)^{r-2}$

$(x-2)^2 r(r-1)(x-2)^{r-2} + 7(x-2)r(x-2)^{r-1} + 9(x-2)^r = 0$

$(x-2)^r[r(r-1) + 7r + 9] = 0$

set $r(r-1) + 7r + 9 = 0$

$r^2 + 6r + 9 = 0$

$(r+3)^2 = 0$

$r = -3$

$$\boxed{y = c_1|x-2|^{-3} + c_2|x-2|^{-3}\ln|x-2|}$$

$\boxed{30}$ 7. Let $A = \begin{bmatrix} 2 & \alpha \\ 1 & 0 \end{bmatrix}$.

(a) Determine conditions on α which ensure the matrix A has

- two real eigenvalues
- no real eigenvalues

Characteristic eqn. $\det(A-\lambda I)=0$

$$\Rightarrow \quad \begin{vmatrix} 2-\lambda & \alpha \\ 1 & -\lambda \end{vmatrix} = 0$$

$$-\lambda(2-\lambda) - \alpha \cdot 1 = 0$$

$$\lambda^2 - 2\lambda - \alpha = 0$$

$$\lambda = \frac{2 \pm \sqrt{4+4\alpha}}{2}$$

$$= 1 \pm \sqrt{1+\alpha}$$

so two (distinct) real eigenvalues when $\alpha > -1$ and none when $\alpha < -1$.

(b) In the case $\alpha = 3$, find two linearly independent eigenvectors of A.

$\alpha = 3 \Rightarrow \lambda = 1 \pm \sqrt{1+3} = -1, 3.$

$\lambda = -1$: $\begin{pmatrix} 3 & 3 \\ 1 & 1 \end{pmatrix} \vec{v} = \vec{0} \Rightarrow \vec{v} = \begin{pmatrix} 1 \\ -1 \end{pmatrix}$ is an eigenvector

$\lambda = 3$: $\begin{pmatrix} -1 & 3 \\ 1 & -3 \end{pmatrix} \vec{v} = \vec{0} \Rightarrow \vec{v} = \begin{pmatrix} 3 \\ 1 \end{pmatrix}$ " " "

Eigen~~values~~ectors corresponding to distinct eigenvalues are linearly independent, so $\left\{ \begin{pmatrix} 1 \\ -1 \end{pmatrix}, \begin{pmatrix} 3 \\ 1 \end{pmatrix} \right\}$ are.

$\boxed{20}$ 8. (a) Solve the matrix equation:

$$\begin{pmatrix} 2 & -2 & -3 \\ 1 & 6 & 2 \\ 4 & 6 & -1 \end{pmatrix} \vec{v} = \begin{pmatrix} 0 \\ 0 \\ 0 \end{pmatrix}$$

$R_1 \leftrightarrow R_2$ $\begin{pmatrix} 1 & 6 & 2 \\ 2 & -2 & -3 \\ 4 & 6 & 1 \end{pmatrix}$

$\begin{matrix} R_2 - 2R_1 \\ R_3 - 4R_1 \end{matrix}$ $\begin{pmatrix} 1 & 6 & 2 \\ 0 & -14 & -7 \\ 0 & -18 & -9 \end{pmatrix}$

$\begin{matrix} -\frac{1}{7} \times R_2 \\ -\frac{1}{9} \times R_3 \end{matrix}$ $\begin{pmatrix} 1 & 6 & 2 \\ 0 & 2 & 1 \\ 0 & 2 & 1 \end{pmatrix}$

$\begin{matrix} R_1 - 3R_2 \\ R_3 - R_2 \end{matrix}$ $\begin{pmatrix} 1 & 0 & -1 \\ 0 & 2 & 1 \\ 0 & 0 & 0 \end{pmatrix}$

$\frac{1}{2} R_2$ $\begin{pmatrix} 1 & 0 & -1 \\ 0 & 1 & 1/2 \\ 0 & 0 & 0 \end{pmatrix}$

By Gaussian elimination, an equivalent equation is

$$\begin{pmatrix} 1 & 0 & -1 \\ 0 & 1 & 1/2 \\ 0 & 0 & 0 \end{pmatrix} \vec{v} = \vec{0}$$

thus let v_3 be arbitrary, say $v_3 = c$

then $v_1 - v_3 = 0 \Rightarrow v_1 = c$

$v_2 + \frac{1}{2} v_3 = 0 \Rightarrow v_2 = -\frac{1}{2} c$

so $\vec{v} = \begin{pmatrix} c \\ -\frac{1}{2}c \\ c \end{pmatrix} = c \cdot \begin{pmatrix} 1 \\ -1/2 \\ 1 \end{pmatrix}$

(b) What does your solution tell you about the columns of the matrix $\begin{pmatrix} 2 & -2 & -3 \\ 1 & 6 & 2 \\ 4 & 6 & -1 \end{pmatrix}$?

Since the homogeneous equation has a nonzero solution, the columns of the matrix are linearly dependent.