Row Reduction

To solve

$$\begin{pmatrix} 1 & 1 & -1 \\ 1 & 3 & 0 \\ -2 & -3 & 2 \end{pmatrix} \vec{x} = \begin{pmatrix} 3 \\ -1 \\ 2 \end{pmatrix}$$

we construct an *augmented* matrix

$$\left(\begin{array}{ccc|c} 1 & 1 & -1 & 3 \\ 1 & 3 & 0 & -1 \\ -2 & -3 & 2 & 2 \end{array}\right)$$

Reduce to upper triangular form.

Step 1: Use row 1 to clear column 1, rows ≥ 2

$$\begin{array}{c} r_1 \\ r_2 - r_1 \\ r_3 + 2r_1 \end{array} \qquad \left(\begin{array}{ccc|c} 1 & 1 & -1 & 3 \\ 0 & 2 & 1 & -4 \\ 0 & -1 & 0 & 8 \end{array}\right)$$

Step 2: Use row 2 to clear column 2, rows ≥ 3

$$\begin{array}{c} r_1 \\ r_2 \\ r_3 + r_2/2 \end{array} \qquad \left(\begin{array}{ccc|c} 1 & 1 & -1 & 3 \\ 0 & 2 & 1 & -4 \\ 0 & 0 & 1/2 & 6 \end{array}\right)$$

If you have a larger matrix keep going. Step 3: Use row 3 to clear column 3, row ≥ 4 etc.

This yields the *triangular* equation

$$\begin{pmatrix} 1 & 1 & -1 \\ 0 & 2 & 1 \\ 0 & 0 & 1/2 \end{pmatrix} \begin{pmatrix} x_1 \\ x_2 \\ x_3 \end{pmatrix} = \begin{pmatrix} 3 \\ -4 \\ 6 \end{pmatrix}$$

which solves as $x_3 = 12$, $x_2 = \frac{-4-x_3}{2} = -8$, $x_1 = 3 - x_2 - (-1)x_3 = 3 + 8 + 12 = 23$. So

$$\vec{x} = \begin{pmatrix} 23 \\ -8 \\ 12 \end{pmatrix}$$

is the solution.

Finding Eigenvalues and Eigenvectors

To find eigenvectors of $A = \begin{pmatrix} 1 & 0 & 0 \\ 2 & 1 & -2 \\ 3 & 2 & 1 \end{pmatrix}$, we try to solve $(A - \lambda I)\vec{x} = -\vec{0}$ by row reduction for eigenvalues λ. Thus we shall consider the augmented matrix

$$\left(\begin{array}{ccc|c} 1-\lambda & 0 & 0 & 0 \\ 2 & 1-\lambda & -2 & 0 \\ 3 & 2 & 1-\lambda & 0 \end{array}\right)$$

First we need to find the values of λ by solving

$$\det(A - \lambda I) = 0.$$

Here

$$\det(A - \lambda I) = +(1-\lambda)\det\begin{pmatrix} 1-\lambda & -2 \\ 2 & 1-\lambda \end{pmatrix} - (0)\det\begin{pmatrix} 2 & -2 \\ 3 & 1-\lambda \end{pmatrix} + (0)\begin{pmatrix} 2 & 1-\lambda \\ 3 & 2 \end{pmatrix} = (1-\lambda)(\lambda^2 - 2\lambda + 5)$$

So $\lambda = 1, 1+2i, 1-2i$.

To find the eigenvector associated to $\lambda = 1$ we consider

$$\left(\begin{array}{ccc|c} 0 & 0 & 0 & 0 \\ 2 & 0 & -2 & 0 \\ 3 & 2 & 0 & 0 \end{array}\right)$$

$$\begin{array}{c} r_3 \\ r_2 \\ r_1 \end{array} \quad \left(\begin{array}{ccc|c} 3 & 2 & 0 & 0 \\ 2 & 0 & -2 & 0 \\ 0 & 0 & 0 & 0 \end{array}\right)$$

$$\begin{array}{c} r_1 \\ 3r_2 - 2r_1 \\ r_3 \end{array} \quad \left(\begin{array}{ccc|c} 3 & 2 & 0 & 0 \\ 0 & -4 & -2 & 0 \\ 0 & 0 & 0 & 0 \end{array}\right)$$

Recall the reordering of rows, this implies $3x_3 + 2x_2 = 0$ and $-4x_2 - 2x_1 = 0$. We'll let $x_2 = \alpha$ be arbitrary. The eigenvectors are then

$$\vec{x} = \alpha \begin{pmatrix} -2 \\ 1 \\ -2/3 \end{pmatrix}$$

The other eigenvalues are similar.