8/8/17

SOLUTIONS

Your name:

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**Math 22 Summer 2017, Midterm 2, Tues Aug 8**

*Please show your work. No credit is given for solutions without work or justification.*

1. [6 points]

3 pts (a) Compute the determinant of $\begin{bmatrix} \cancel{0} & 2 & 0 & 1 \\ \cancel{0} & -1 & 0 & 2 \\ \cancel{2} & \cancel{21} & \cancel{7} & \cancel{4} \\ \cancel{0} & 3 & 1 & 5 \end{bmatrix}$.

there are two options for a row or col. with only one nonzero entry.

Cofactor expansion.

$$\det A = +2 \begin{vmatrix} 2 & 0 & 1 \\ -1 & 0 & 2 \\ 3 & 1 & 5 \end{vmatrix}$$

sign for cofactor expansion is $-$.

$$= (+2)(-1) \underbrace{\begin{vmatrix} 2 & 1 \\ -1 & 2 \end{vmatrix}}_{2(2)-1(-1)} = 2(-1)5 = -10.$$

Swapping

3 pts (b) Let $A_0$ be an invertible $4 \times 4$ matrix with $\det A_0 = 1$, and suppose:

- $A_1$ is obtained from $A_0$ by interchanging 2 rows,
- $A_2$ is obtained from $A_1$ (note: not $A_0$) by scaling a row of $A_1$ by 3,
- $A_3$ is obtained from $A_2$ (note: not $A_0$) by row-replacement.

Find the determinants of these matrices and fill them in below:

$\det A_1 = -1$ ← since $\det A_0 = 1$

$\det A_2 = -3$ scaling

$\det A_3 = -3$ row replacement has no effect.

1 pt each & don't penalize cumulative errors

$$\begin{array}{cccc} & x_2 \text{ free} & & x_4 \text{ free} \\ & \downarrow & & \downarrow \end{array}$$

2. [8 points] Let $A = \begin{bmatrix} -1 & 2 & -6 & -3 \\ 2 & -4 & 7 & 6 \\ -1 & 2 & -3 & -3 \end{bmatrix}$ which is row-equivalent to $\begin{bmatrix} \boxed{1} & -2 & 0 & 3 \\ 0 & 0 & \boxed{1} & 0 \\ 0 & 0 & 0 & 0 \end{bmatrix}$.

2 pts. (a) Find a basis for Col $A$. ← is in REF (good)

Take pivot columns of original matrix $A$:

$$\left\{ \begin{bmatrix} -1 \\ 2 \\ -1 \end{bmatrix}, \begin{bmatrix} -6 \\ 7 \\ -3 \end{bmatrix} \right\}$$

3 pts (b) Find a basis for Nul $A$. use param. vec. form of REF:

$$\left.\begin{aligned} x_1 &= 2x_2 - 3x_4 \\ x_2 &= x_2 \\ x_3 &= 0 \\ x_4 &= x_4 \end{aligned}\right\} \quad \vec{x} = \begin{bmatrix} 2 \\ 1 \\ 0 \\ 0 \end{bmatrix} x_2 + \begin{bmatrix} -3 \\ 0 \\ 0 \\ 1 \end{bmatrix} x_4 \qquad \text{(parametrizes Nul } A\text{)}$$

$$\Rightarrow \text{basis is } \left\{ \begin{bmatrix} 2 \\ 1 \\ 0 \\ 0 \end{bmatrix}, \begin{bmatrix} -3 \\ 0 \\ 0 \\ 1 \end{bmatrix} \right\}$$

(c) Prove that the null space of any $3 \times 4$ matrix $A$ is a subspace of $\mathbb{R}^4$.

This is a standard proof that works for any $\overset{3}{m} \times \overset{4}{n}$ matrix $A$:

Nul $A$ is a subset of $\mathbb{R}^4$ since, by definition, Nul $A = \{\vec{x} \in \mathbb{R}^4 : A\vec{x} = \vec{0}\}$

Test 3 subspace axioms:

a) $\vec{0} \in$ Nul $A$ since $A\vec{0} = \vec{0}$.

b) Let $\vec{x}, \vec{y} \in$ Nul $A$. then $A\vec{x} = \vec{0}$ and $A\vec{y} = \vec{0}$

Adding the equations, $A\vec{x} + A\vec{y} \underset{\text{by linearity of mat-vec prod.}}{=} A(\vec{x} + \vec{y}) = \vec{0} + \vec{0} = \vec{0}$

So $\vec{x} + \vec{y} \in$ Nul $A$

c) Let $\vec{x} \in$ Nul $A$, $c \in \mathbb{R}$. then $A(c\vec{x}) \overset{\text{linearity}}{=} c(A\vec{x}) = c\vec{0} = \vec{0}$

So $c\vec{x} \in$ Nul $A$ ☒

3. [6 points]

here, a proof they are not L. I. could also be giving $\vec{v}_1 + \vec{v}_2 - \vec{v}_3 = \vec{0}$, a dependence relation (*)

3pts (a) Let $H = \left\{ \begin{bmatrix} x_1 \\ x_2 \\ x_3 \end{bmatrix} \in \mathbb{R}^3 : x_1 + x_2 + x_3 = 0 \right\}$.

Is the set $\left\{ \begin{bmatrix} 1 \\ -1 \\ 0 \end{bmatrix}, \begin{bmatrix} 0 \\ 1 \\ -1 \end{bmatrix}, \begin{bmatrix} 1 \\ 0 \\ -1 \end{bmatrix} \right\}$ a basis for $H$? Explain.

$\nwarrow \vec{v}_1 \quad \nwarrow \vec{v}_2 \quad \nwarrow \vec{v}_3$

1pt for not a basis, 2 for why

Although $H = \text{Span}\{\vec{v}_1, \vec{v}_2, \vec{v}_3\}$ (see below), the set is not linearly independent, so is not a basis.

$$[\vec{v}_1 \vec{v}_2 \vec{v}_3] = \begin{bmatrix} 1 & 0 & 1 \\ -1 & 1 & 0 \\ 0 & -1 & -1 \end{bmatrix} \sim \begin{bmatrix} \boxed{1} & 0 & 1 \\ 0 & \boxed{1} & 1 \\ 0 & 0 & 0 \end{bmatrix}$$

in REF, showing lack of linear independence.

free variable

Not needed:
To check span claim: $H = \text{Nul}\begin{bmatrix} 1 & 1 & 1 \end{bmatrix} = \text{Span}\left\{ \begin{bmatrix} -1 \\ 1 \\ 0 \end{bmatrix}, \begin{bmatrix} -1 \\ 0 \\ 1 \end{bmatrix} \right\}$

which you can see equals $\text{Span}\{\vec{v}_1, \vec{v}_3\} = \text{Span}\{\vec{v}_1, \vec{v}_2, \vec{v}_3\}$ via * above.

3pts (b) Let $A$ and $B$ be matrices such that $AB$ exists. Prove that rank $(AB) \le$ rank $A$. [Hint: each column of $AB$ is in Col $A$.]

don't need for full points; can assume true.

Why is hint true? since $j^{th}$ col of $(AB)$ is $A$ times ($j^{th}$ col of $B$).

So, Col $(AB)$ is the span of vectors all of which lie in Col $A$.

thus $\boxed{\text{Col}(AB) \subseteq \text{Col}\, A}$ +1 ← subset relation.

Since Col $(AB)$ is a subspace (because it's a span), it is then a subspace of Col $A$.

+1

Thm 11 in Ch. 4 can then apply: $\boxed{\dim \text{Col}(AB) \le \dim \text{Col}\, A}$

("dimension of a subspace can't exceed dimension of the V.S. it's a subspace of")

By definition of rank (= dim Col), the claim follows. ////



this can be proved via spanning set thm applied to a basis for the V.S. (not needed)

4. [9 points] Let $A = \begin{bmatrix} 5 & -4 & -2 \\ 2 & -1 & -2 \\ 0 & 0 & 3 \end{bmatrix}$. [Hint: numbers will come out very simply, so stop and check your work if they are not!]

3pts (a) Use the characteristic polynomial to find $A$'s eigenvalues and their algebraic multiplicies:

$$\det(A-\lambda I) = \begin{vmatrix} 5-\lambda & -4 & -2 \\ 2 & -1-\lambda & -2 \\ 0 & 0 & 3-\lambda \end{vmatrix} = (3-\lambda)\underbrace{\begin{vmatrix} 5-\lambda & -4 \\ 2 & -1-\lambda \end{vmatrix}}_{\lambda^2-4\lambda+3 \,=\, (\lambda-3)(\lambda-1)}$$

cofactor about this.

So, char. poly factors as $(3-\lambda)^2(1-\lambda)$ $\Rightarrow$ Eigvals: $\lambda=3$ has multiplicity 2.
$\lambda = 1$ " " 1.

4pts (b) For each distinct eigenvalue of $A$, find a basis for its eigenspace:

2pts each

$\boxed{\lambda_1 = 1:}$ $A - 1I = \begin{bmatrix} 4 & -4 & -2 \\ 2 & -2 & -2 \\ 0 & 0 & 2 \end{bmatrix} \sim \begin{bmatrix} 1 & -1 & -1 \\ 0 & 0 & 1 \\ 0 & 0 & 0 \end{bmatrix}$ EF

$\sim \begin{bmatrix} \boxed{1} & -1 & 0 \\ 0 & 0 & \boxed{1} \\ 0 & 0 & 0 \end{bmatrix}$ REF so $\vec{v}_1 = \begin{bmatrix} 1 \\ 1 \\ 0 \end{bmatrix}$; $\{\vec{v}_1\}$ is basis for eigenspace.

free

$\boxed{\lambda_2 = 3:}$ $A - 3I = \begin{bmatrix} 2 & -4 & -2 \\ 2 & -4 & -2 \\ 0 & 0 & 0 \end{bmatrix} \sim \begin{bmatrix} \boxed{1} & -2 & -1 \\ 0 & 0 & 0 \\ 0 & 0 & 0 \end{bmatrix} \Rightarrow \begin{cases} x_1 = 2x_2 + x_3 \\ x_2 = x_2 \\ x_3 = x_3 \end{cases}$

so $\left\{ \begin{bmatrix} 2 \\ 1 \\ 0 \end{bmatrix}, \begin{bmatrix} 1 \\ 0 \\ 1 \end{bmatrix} \right\}$ is basis for eigenspace.

using eigenvector eqn. $A\vec{v}_1 = \lambda_1 \vec{v}_1$

2pts (c) Evaluate $A^{2017}\begin{bmatrix} 1 \\ 1 \\ 0 \end{bmatrix}$ $= \lambda_1^{2017}\vec{v}_1 = 1^{2017}\vec{v}_1 = \vec{v}_1 = \begin{bmatrix} 1 \\ 1 \\ 0 \end{bmatrix}$

is $\vec{v}_1$ above.

5. [8 points]

ie coefficients   m   n

(a) A linear system has a system matrix $A$ of size $7 \times 9$ (ie 7 equations in 9 unknowns). Say you know that there is some right-hand side vector for which there is no solution. What is the smallest dim Nul $A$ may be, and why?

3pts.

[1/3 if ignored info about RHS $\vec{b}$]

[2/3 if confused m & n.]

If there is some $\vec{b}$ such that $A\vec{x} = \vec{b}$ inconsistent, there cannot be a pivot in every row. So there are at most 6 $(= m-1)$ pivots. By the rank theorem (or by counting free variables),

$$\dim \text{Nul } A = n - \text{rank } A \geq n - 6 = 3.$$

3pts

(b) Now let $A$ be any matrix. If the system $A\mathbf{x} = \mathbf{b}$ is consistent for all right-hand sides $\mathbf{b}$, explain why the system $A^T\mathbf{x} = \mathbf{0}$ has only the trivial solution.

$A$ is $m \times n$:

$A\vec{x} = \vec{b}$ consistent for all $\vec{b}$ $\Rightarrow$ rank $A = m$ (ie, pivot in every row).

Since rank $A^T$ = rank $A$ (since one is dim Row $A$, the other dim Col $A$),

we get rank $A^T = m$   note: pivot positions are not transposed!

$A^T$

Since $A^T$ has $m$ columns, it has a pivot in every column, ie no free variables, so $A^T\vec{x} = \vec{0} \Rightarrow \vec{x} = \vec{0}$, unique.

2pts.

if claims case $A$ = zero matrix, no non-empty subset is basis, get 2pts.

(c) Let $A$ be any matrix. Is some subset of the rows of a $A$ a basis for Row $A$ ? Prove your answer. [As always, indicate what if any theorem(s) you use.]

Ie, does there exist a subset of rows of $A$ that are basis for Row $A$?

- You cannot appeal to taught procedure for a basis for Row $A$, since this is the set of pivot rows in the row-reduced $A$, which differ in general from any rows of the original $A$!

Proof 1) let $\vec{r}_1 \cdots \vec{r}_m$ be the rows of $A$. By defn, Row $A$ = Span$\{\vec{r}_1 \cdots \vec{r}_m\}$. By the Spanning set theorem (Thm 5, Ch.4), some subset of the elements $\{\vec{r}_1, \cdots \vec{r}_m\}$ is a basis for their span. If $A$ is 0 matrix, empty subset is a basis. $\square$

OR Proof 2) Row $A$ = Col $A^T$ so pivot cols of $A^T$ are basis for Row $A$.

6. [6 points]

call $\mathcal{B}$

3 pts (a) Is the set $\{1+t, 1-t, t+2t^2\}$ a basis for $\mathbb{P}_2$? Prove your answer. [State any theorems or results that you use.]

$\mathbb{P}_2$ is isomorphic to $\mathbb{R}^3$, so we can answer via the set's coord vectors (via the std. basis $\{1, t, t^2\} = S$, coord map)

[1 pt was for isomorphism]

$1+t \xrightarrow[{[\cdot]_S}]{\text{coord map}} \begin{bmatrix} 1 \\ 1 \\ 0 \end{bmatrix}$ $\longrightarrow$ is basis for $\mathbb{R}^3$?

$1-t \longrightarrow \begin{bmatrix} 1 \\ -1 \\ 0 \end{bmatrix}$

$t+2t^2 \longrightarrow \begin{bmatrix} 0 \\ 1 \\ 2 \end{bmatrix}$

Stack & reduce:

$$\begin{bmatrix} 1 & 1 & 0 \\ 1 & -1 & 1 \\ 0 & 0 & 2 \end{bmatrix} \sim \begin{bmatrix} 1 & 1 & 0 \\ 0 & -2 & 1 \\ 0 & 0 & 1 \end{bmatrix}$$

full set of $n=3$ pivots.

$\Rightarrow$ by IMT, is a basis for $\mathbb{R}^3$

$\Rightarrow$ by isomorphism, $\mathcal{B}$ was basis for $\mathbb{P}_2$.

a.k.a. coordinates.

3 pts (b) Find the coefficients of $4(1+t)^2$ relative to the set from part (a).

which we now know is a basis

expand.

$p(t) = 4 + 8t + 4t^2$

so $[p(t)]_S = \begin{bmatrix} 4 \\ 8 \\ 4 \end{bmatrix}$

By isomorphism, can find coeffs via a linear system in $\mathbb{R}^3$:

$$\left[\begin{array}{ccc|c} 1 & 1 & 0 & 4 \\ 1 & -1 & 1 & 8 \\ 0 & 0 & 2 & 4 \end{array}\right] \sim \begin{bmatrix} 1 & 1 & 0 & 4 \\ 0 & -2 & 1 & 4 \\ 0 & 0 & 1 & 2 \end{bmatrix} \sim \begin{bmatrix} 1 & 1 & 0 & 4 \\ 0 & 2 & 0 & -2 \\ 0 & 0 & 1 & 2 \end{bmatrix}$$

$$\sim \left[\begin{array}{ccc|c} 1 & 0 & 0 & 5 \\ 0 & 1 & 0 & -1 \\ 0 & 0 & 1 & 2 \end{array}\right]$$

so coeffs are $\begin{bmatrix} 5 \\ -1 \\ 2 \end{bmatrix}$

read off $\vec{c} = \begin{bmatrix} c_1 \\ c_2 \\ c_3 \end{bmatrix}$

7. [7 points] In this question only, no working is needed; just circle T or F.

(a) T / F: The eigenvalues of a lower-triangular matrix (ie, all zeros above the diagonal) are the diagonal entries.

$$\begin{bmatrix} a_{11}-\lambda & 0 & 0 & \cdots \\ a_{21} & a_{22}-\lambda & 0 & \cdots \\ a_{31} & a_{32} & a_{33}-\lambda & \\ \vdots & \vdots & & \ddots \end{bmatrix}$$

char poly $= (a_{11}-\lambda)(a_{22}-\lambda)\cdots$

(b) T / F: The dimension of an eigenspace can never exceed the algebraic multiplicity of the corresponding eigenvalue.

See sec. 5.2.

(c) T / F: An eigenvector with eigenvalue 2 could be a linear combination of an eigenvector with eigenvalue 3 and an eigenvector with eigenvalue 4.

assume these come from same matrix.

In 5.1 it's proved that eigvecs from distinct eigenspaces (different $\lambda$'s) are L.I.

(d) T / F: Row reduction of a matrix always preserves its row space.

since Row $B \subseteq$ Row $A$ & Row $A \subseteq$ Row $B$ where $B \sim A$.

(e) T / F: Row reduction of a square matrix always preserves its eigenvalues.

R.R. was never used to find eigenvalues.

(f) T / F: If $T : \mathbb{R}^5 \to \mathbb{R}^5$ is a linear transformation with standard matrix $A$ and the rank of $A$ is 2, then it is possible to have $T(\mathbf{x}) = \mathbf{0}$ for every $\mathbf{x}$ in the domain.

$\hookrightarrow$ 2 pivots. $\hookrightarrow$ would imply $A =$ zero matrix.

(g) T / F: Suppose $T : \mathbb{R}^n \to \mathbb{R}^m$ is an isomorphism from $\mathbb{R}^n$ to $\mathbb{R}^m$, with $n, m > 0$, with standard matrix $A$. Then it is impossible for Nul $A$ and Col $A$ to have the same dimension.

isomorphism means same dimension, so $m = n$,
and one-to-one, so dim Nul $A = 0$,
and onto, so dim Col $A = n$
but $n \neq 0$.

So, indeed it is impossible.