A) Find det of $\begin{bmatrix} 0 & 1 & 0 & 1 \\ 0 & 0 & 5 & 2 \\ 2 & 7 & -2 & 3 \\ 0 & 0 & 2 & -3 \end{bmatrix}$

B) Find eigenvalues and bases for corresponding eigenspaces of

i) $\begin{bmatrix} 7 & 2 \\ -4 & 1 \end{bmatrix}$

ii) $\begin{bmatrix} -1 & 0 & 1 \\ -3 & 4 & 1 \\ 0 & 0 & 2 \end{bmatrix}$ [Hint: -1 is an eigenvalue]

PTO.

C) Let λ be an eigenvalue of an invertible matrix A. Show that λ^{-1} is an eigenvalue of A^{-1} [Hint: write definition of $\vec{x}$ being eigenvector of A w/ eigenvalue λ].

D) Find bases for Col A, Nul A and Row A for

$$A = \begin{bmatrix} 3 & 1 & 0 \\ 0 & 2 & 6 \\ 6 & 3 & 3 \end{bmatrix}$$

E) Is the set $W = \left\{ \begin{bmatrix} x \\ y \\ z \end{bmatrix} : x \geq 0 \right\}$ a subspace of $\mathbb{R}^3$? (explain).

F) If zero is an eigenvalue of A^T, how many solutions to $A\vec{x} = \vec{b}$ are there? (A is $n \times n$)

BONUS: in F) say $\lambda = 0$ has multiplicity p, what is dim Nul A ?

MATH 22 : Midterm 2 practice. ~> SOLUTIONS ~

A) Find det of $\begin{bmatrix} 0 & 1 & 0 & 1 \\ 0 & 0 & 5 & 2 \\ 2 & 7 & -2 & 3 \\ 0 & 0 & 2 & -3 \end{bmatrix}$

a good col to expand along: cofactors

$$\det = 2 \cdot \underbrace{\begin{vmatrix} 1 & 0 & 1 \\ 0 & 5 & 2 \\ 0 & 2 & -3 \end{vmatrix}}_{1 \cdot \begin{vmatrix} 5 & 2 \\ 2 & -3 \end{vmatrix} = -19}$$

$\det = -38.$

B) Find eigenvalues and bases for corresponding eigenspaces of

i) $\begin{bmatrix} 7 & 2 \\ -4 & 1 \end{bmatrix}$ $\quad (7-\lambda)(1-\lambda)+8 = \lambda^2 - 8\lambda + 15 = (\lambda-5)(\lambda-3) \quad \lambda = 3, 5.$

$\lambda_1 = 3$: $A - 3I$: $\begin{bmatrix} 4 & 2 \\ -4 & -2 \end{bmatrix} \sim \begin{bmatrix} 2 & 1 \\ 0 & 0 \end{bmatrix}$ so $x_1 = -\frac{1}{2}x_2$, x_2 = free. $\vec{v}_1 = \begin{bmatrix} -1 \\ 2 \end{bmatrix}$

$\lambda_2 = 5$: $\begin{bmatrix} 2 & 2 \\ -4 & -4 \end{bmatrix} \sim \begin{bmatrix} 1 & 1 \\ 0 & 0 \end{bmatrix}$ so $\vec{v}_2 = \begin{bmatrix} 1 \\ -1 \end{bmatrix}$

ii) $\begin{bmatrix} -1 & 0 & 1 \\ -3 & 4 & 1 \\ 0 & 0 & 2 \end{bmatrix}$ [Hint: -1 is an eigenvalue]

use for cofactor expansion:

$$(2-\lambda)\left[(-1-\lambda)(4-\lambda) - 0(-3)\right] = (2-\lambda)(-1-\lambda)(4-\lambda)$$

3 distinct simple eigvals $\Rightarrow$ only can be one L.I. eigvec for each.

$\lambda_1 = -1$: $\begin{matrix} 0 & 0 & 1 \\ -3 & 5 & 1 \\ 0 & 0 & 3 \end{matrix} \sim \begin{matrix} \boxed{1} & -5/3 & 0 \\ 0 & 0 & \boxed{1} \\ 0 & 0 & 0 \end{matrix}$ $\leftarrow$ use 1st row: so $\vec{v}_1 = \begin{bmatrix} 5/3 \\ 1 \\ 0 \end{bmatrix}$

↑ free

param vec. form for $(A-\lambda I)\vec{x} = \vec{0}$ soln. set.

$\lambda_2 = 2$: $\begin{matrix} -3 & 0 & 1 \\ -3 & 2 & 1 \\ 0 & 0 & 0 \end{matrix} \sim \begin{matrix} 1 & 0 & -1/3 \\ 0 & 1 & 0 \\ 0 & 0 & 0 \end{matrix}$ $\quad \vec{v}_2 = \begin{bmatrix} 1/3 \\ 0 \\ 1 \end{bmatrix}$

can tell singular

↑ free

$\lambda_3 = 4$: $\begin{matrix} -5 & 0 & 1 \\ -3 & 0 & 1 \\ 0 & 0 & -3 \end{matrix} \sim \begin{matrix} 1 & 0 & 0 \\ 0 & 0 & 1 \\ 0 & 0 & 0 \end{matrix}$ $\quad \vec{v}_3 = \begin{bmatrix} 0 \\ 1 \\ 0 \end{bmatrix}$

can tell singular

↑ free

PTO.

c) Let λ be an eigenvalue of an invertible matrix A.
Show that λ^{-1} is an eigenvalue of A^{-1} [Hint: write definition of $\vec{x}$ being eigenvector of A w/ eigenvalue λ]

$A\vec{x} = \lambda\vec{x}$ & have A^{-1} exists, so hit it from left:

$\underbrace{A^{-1}A}_{I}\vec{x} = A^{-1}\lambda\vec{x}$ so $\vec{x} = \lambda A^{-1}\vec{x}$ so $\lambda^{-1}\vec{x} = A^{-1}\vec{x}$ which is eigval. equation for A^{-1}.

can invert λ since all eigvals $\neq 0$ of A.

Note eigenvecs are same!

D) Find bases for Col A, Nul A and Row A for

$$A = \begin{bmatrix} 3 & 1 & 0 \\ 0 & 2 & 6 \\ 6 & 3 & 3 \end{bmatrix} \sim \begin{bmatrix} 3 & 1 & 0 \\ 0 & 2 & 6 \\ 0 & 1 & 3 \end{bmatrix} \sim \underset{EF}{\begin{bmatrix} \boxed{3} & 1 & 0 \\ 0 & \boxed{1} & 3 \\ 0 & 0 & 0 \end{bmatrix}} \sim \begin{bmatrix} 3 & 0 & -3 \\ & 1 & 3 \\ & & \end{bmatrix} \sim \underset{REF}{\begin{bmatrix} 1 & 0 & -1 \\ & 1 & 3 \\ & & \end{bmatrix}}$$

piv. cols. free

basis Col A : $\begin{bmatrix} 3 \\ 0 \\ 6 \end{bmatrix}, \begin{bmatrix} 1 \\ 2 \\ 3 \end{bmatrix}$ piv. cols in original A.

param vec from REF:

$x_1 = +x_3$
$x_2 = -3x_3$
$x_3 = x_3$

so $\vec{x} = \begin{bmatrix} 1 \\ -3 \\ 1 \end{bmatrix} x_3$

basis Row A : $(3,1,0), (0,1,3)$
pivot rows of EF for A

basis Nul A = $\begin{bmatrix} 1 \\ -3 \\ 1 \end{bmatrix}$

E) Is the set $W = \left\{ \begin{bmatrix} x \\ y \\ z \end{bmatrix} : x \geq 0 \right\}$ a subspace of $\mathbb{R}^3$? (explain).

is $\vec{0}$ in W, yes, & closed under addition.
But if $\vec{x}$ in W, $c\vec{x}$ not in W for scalar $c < 0$ $\Rightarrow$ No.

F) <u>If zero is an eigenvalue of A^T</u>, how many solutions to $A\vec{x} = \vec{b}$ are there? (A is $n \times n$)

means $A^T\vec{x} = 0 \cdot \vec{x}$ for nonzero $\vec{x}$,
ie A^T not invertible.

By IMT, A not invertible, so there are either <u>zero</u> or <u>∞</u> number of solns.

<u>Bonus:</u> If we knew $\lambda = 0$ has multiplicity $p \geq 1$, then $\dim \text{Nul} A = \dim \text{Nul} A^T = p$.
by equality of ranks; something you'd want to show.