MATH 22 LECTURE 29 CLASSWORK

AUGUST 23, 2017

(1) Let $A = \begin{bmatrix} -1 & 0 \\ -1 & 0 \end{bmatrix}$.

(a) Maximize $\|A\mathbf{x}\|$ subject to the constraint that $\|\mathbf{x}\| = 1$.

$A^TA = \begin{bmatrix} 2 & 0 \\ 0 & 0 \end{bmatrix}$, so $\lambda_1 = 2$, $\lambda_2 = 0 \Rightarrow \sigma_1 = \sqrt{2}$

$\sqrt{2}$ is the largest $\|A\vec{x}\|$ can be for $\vec{x}$ on the unit circle in $\mathbb{R}^2$.

(b) Compute the SVD of A and A^T.

$$A = \underbrace{\begin{bmatrix} -1/\sqrt{2} & -1/\sqrt{2} \\ -1/\sqrt{2} & 1/\sqrt{2} \end{bmatrix}}_{U,\ \text{columns } \vec{u}_1,\ \vec{u}_2} \underbrace{\begin{bmatrix} \sqrt{2} & 0 \\ 0 & 0 \end{bmatrix}}_{\Sigma} \underbrace{\begin{bmatrix} 1 & 0 \\ 0 & 1 \end{bmatrix}}_{V,\ \text{columns } \vec{v}_1,\ \vec{v}_2}^T$$

$A^T = V\Sigma^T U^T$

(c) Find orthonormal bases for as many fundamental spaces as possible!

$\{\vec{u}_1\}$ is an o.n.b. for $\text{Col } A = (\text{Nul}(A^T))^\perp$

$\{\vec{u}_2\}$ is an o.n.b. for $(\text{Col } A)^\perp = \text{Nul}(A^T)$

$\{\vec{v}_2\}$ is an o.n.b. for $\text{Nul } A = (\text{Row } A)^\perp$

$\{\vec{v}_1\}$ is an o.n.b. for $(\text{Nul } A)^\perp = \text{Row } A$

(d) What is the best rank 1 approximation of A? A is rank 1, so A.

(2) Let $A = \begin{bmatrix} 1 & 2 & 0 \\ 1 & 0 & 2 \\ 0 & 1 & -1 \end{bmatrix}$, $\mathbf{b} = \begin{bmatrix} 2 \\ 0 \\ -5 \end{bmatrix}$.

(a) Find all least-squares solutions to $A\mathbf{x} = \mathbf{b}$.

$$A^TA = \begin{bmatrix} 2 & 2 & 2 \\ 2 & 5 & -1 \\ 2 & -1 & 5 \end{bmatrix}, \quad A^T\vec{b} = \begin{bmatrix} 2 \\ -1 \\ 5 \end{bmatrix}$$

$$\begin{bmatrix} 2 & 2 & 2 & 2 \\ 2 & 5 & -1 & -1 \\ 2 & -1 & 5 & 5 \end{bmatrix} \sim \begin{bmatrix} 1 & 0 & 2 & 2 \\ 0 & 1 & -1 & -1 \\ 0 & 0 & 0 & 0 \end{bmatrix}$$

$$\Rightarrow \hat{x} = \begin{bmatrix} 2 \\ -1 \\ 0 \end{bmatrix} + x_3 \begin{bmatrix} -2 \\ 1 \\ 1 \end{bmatrix}, \text{ where } x_3 \in \mathbb{R}$$

(b) Let $W = \text{Col}A$. Decompose $\mathbf{b} = \hat{\mathbf{b}} + \mathbf{z}$ with $\hat{\mathbf{b}} \in W$ and $\mathbf{z} \in W^{\perp}$. Prove that this decomposition is unique.

$$\hat{b} = A\hat{x} = A\begin{bmatrix} 2 \\ -1 \\ 0 \end{bmatrix} + x_3 A \begin{bmatrix} -2 \\ 1 \\ 1 \end{bmatrix} = \begin{bmatrix} 0 \\ 2 \\ -1 \end{bmatrix}$$

$\vec{z} = \vec{b} - \hat{b}$. Suppose $\vec{b} = \hat{b} + \vec{z} = \hat{b}_1 + \vec{z}_1$ with $\hat{b}_1 \in W$ and $\vec{z}_1 \in W^{\perp}$. Then

$\underbrace{\hat{b} - \hat{b}_1}_{\in W} = \underbrace{\vec{z}_1 - \vec{z}}_{\in W^{\perp}}$ which implies

$$\left.\begin{array}{l} (\hat{b} - \hat{b}_1)\cdot(\hat{b} - \hat{b}_1) = 0 \\ (\vec{z}_1 - \vec{z})\cdot(\vec{z}_1 - \vec{z}) = 0 \end{array}\right\} \Rightarrow \begin{array}{l} \hat{b} - \hat{b}_1 = \vec{0} \\ \vec{z}_1 - \vec{z} = \vec{0} \end{array}$$

(c) Can we compute the QR-factorization of A? No. We need A to have linearly independent columns to compute QR.

(3) Let $A = \begin{bmatrix} 4 & -2 \\ 1 & 1 \end{bmatrix}$.

(a) Check that A is diagonalizable and diagonalize it.

Charpoly$(A) = (3-\lambda)(2-\lambda)$, so we know A diagonalizable.

we find $\vec{x}_1 = \begin{bmatrix} 2 \\ 1 \end{bmatrix} \in Nul(A - \lambda_1 I_2)$ $\lambda_1 := 3$

$\vec{x}_2 = \begin{bmatrix} -1 \\ -1 \end{bmatrix} \in Nul(A - \lambda_2 I_2)$ $\lambda_2 := 2$

Then $A = \underbrace{\begin{bmatrix} 2 & -1 \\ 1 & -1 \end{bmatrix}}_{P} \underbrace{\begin{bmatrix} 3 & 0 \\ 0 & 2 \end{bmatrix}}_{D} \underbrace{\begin{bmatrix} 1 & -1 \\ 1 & -2 \end{bmatrix}}_{P^{-1}}$

(b) Is the matrix P unique? If so, prove it. If not, provide an example.

$A = \begin{bmatrix} 1 & 1 \\ 1/2 & 1 \end{bmatrix} \begin{bmatrix} 3 & 0 \\ 0 & 2 \end{bmatrix} \begin{bmatrix} 2 & -2 \\ -1 & 2 \end{bmatrix}$, So No.

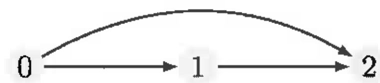
Bases are not unique.

(c) Compute $A^{100} \begin{bmatrix} 1 \\ -1 \end{bmatrix}$. $P := \begin{bmatrix} 2 & -1 \\ 1 & -1 \end{bmatrix}$.

$P^{-1} \begin{bmatrix} 1 \\ -1 \end{bmatrix} = \begin{bmatrix} 2 \\ 3 \end{bmatrix}$, $P \cdot D^{100} = \begin{bmatrix} 2 & -1 \\ 1 & -1 \end{bmatrix} \begin{bmatrix} 3^{100} & 0 \\ 0 & 2^{100} \end{bmatrix} = \begin{bmatrix} 2 \cdot 3^{100} & (-1) 2^{100} \\ 3^{100} & (-1) 2^{100} \end{bmatrix}$

$\Rightarrow \underbrace{P \cdot D^{100} \cdot P^{-1} \begin{bmatrix} 1 \\ -1 \end{bmatrix}}_{A^{100} \begin{bmatrix} 1 \\ -1 \end{bmatrix}} = \underbrace{\begin{bmatrix} 4 \cdot 3^{100} - 3 \cdot 2^{100} \\ 2 \cdot 3^{100} - 3 \cdot 2^{100} \end{bmatrix}}_{\in \mathbb{R}^2}$

(4) Consider the "web" given below:



(a) Use the PageRank algorithm with $\alpha = 1$ to find a probability vector that measures the importance of each node. Explain how the algorithm changes if we instead let $\alpha = 0.85$.

$$S = \begin{bmatrix} 0 & 0 & 1/3 \\ 1/2 & 0 & 1/3 \\ 1/2 & 1 & 1/3 \end{bmatrix}, \quad \text{charpoly} = (\lambda - 1) \cdot \text{quadratic}$$

why?

$$S - I_3 = \begin{bmatrix} -1 & 0 & 1/3 \\ 1/2 & -1 & 1/3 \\ 1/2 & 1 & -2/3 \end{bmatrix} \sim \begin{bmatrix} 1 & 0 & -1/3 \\ 0 & 1 & -1/2 \\ 0 & 0 & 0 \end{bmatrix}$$

$\Rightarrow \lambda = 1$ eigenvector is $\begin{bmatrix} 1/3 \\ 1/2 \\ 1 \end{bmatrix}$.

Rescaling we get the desired probability vector

$$\vec{q} = \left(\frac{1}{11/6}\right) \begin{bmatrix} 1/3 \\ 1/2 \\ 1 \end{bmatrix} = \begin{bmatrix} 2/11 \\ 3/11 \\ 6/11 \end{bmatrix}.$$

If $\alpha = 0.85$, then take $G = \alpha S + (1-\alpha)\left(\frac{1}{2}\right)\begin{bmatrix} 1 & 1 \\ 1 & 1 \end{bmatrix}$ and repeat this process with G.

(b) How do you know that the steady-state vector (of the dynamical system defined by A) is unique?

S^2 has all strictly positive values (i.e. S is regular) which (by a black box) guarantees a unique steady state vector.

Section 4.9.