MATH22 - Linear Algebra with Applications

Exam I ANSWERS

July 11, 2007

1. (30 points) Define each of the following:

(a) a linearly independent set

(b) the span of a set of vectors

(c) an onto function

Answer:

(a) A set of vectors $\{\mathbf{v_1}, \mathbf{v_2}, \dots, \mathbf{v_n}\}$ is linearly independent if the only solution for $(x_1, x_2, \dots, x_n)$ in the equation

$$x_1\mathbf{v_1} + x_2\mathbf{v_2} + \cdots + x_n\mathbf{v_n} = \mathbf{0}$$

is the trivial solution $(x_1, x_2, \dots, x_n) = (0, 0, \dots, 0)$.

(b) The span of a set of vectors is the set of all linear combinations of the vectors in that set.

(c) An onto function is a function whose range and codomain are the same.

2. (30 points) Calculate the inverse of

$$A = \begin{bmatrix} 1 & 0 & 1 \\ 0 & 1 & -1 \\ -1 & 1 & -1 \end{bmatrix}.$$

Answer:

$$\left[\begin{array}{ccc|ccc} 1 & 0 & 1 & 1 & 0 & 0 \\ 0 & 1 & -1 & 0 & 1 & 0 \\ -1 & 1 & -1 & 0 & 0 & 1 \end{array}\right] \rightarrow_{R_1+R_3\rightarrow R_3} \left[\begin{array}{ccc|ccc} 1 & 0 & 1 & 1 & 0 & 0 \\ 0 & 1 & -1 & 0 & 1 & 0 \\ 0 & 1 & 0 & 1 & 0 & 1 \end{array}\right] \rightarrow_{-R_2+R_3\rightarrow R_3}$$

$$\left[\begin{array}{ccc|ccc} 1 & 0 & 1 & 1 & 0 & 0 \\ 0 & 1 & -1 & 0 & 1 & 0 \\ 0 & 0 & 1 & 1 & -1 & 1 \end{array}\right] \rightarrow_{\substack{R_3+R_2\rightarrow R_2 \\ -R_3+R_1\rightarrow R_1}} \left[\begin{array}{ccc|ccc} 1 & 0 & 0 & 0 & 1 & -1 \\ 0 & 1 & 0 & 1 & 0 & 1 \\ 0 & 0 & 1 & 1 & -1 & 1 \end{array}\right],$$

so we have

$$A^{-1} = \begin{bmatrix} 0 & 1 & -1 \\ 1 & 0 & 1 \\ 1 & -1 & 1 \end{bmatrix}.$$

3. (30 points) Solve the following system:

$$\begin{aligned} x_1 \quad\quad\quad +x_3 &= 2 \\ x_2 \; -x_3 &= 1 \\ -x_1 \; +x_2 \; -x_3 &= 0. \end{aligned}$$

(Hint: Save yourself some time and trouble by using the result of #2.)

Answer:

This system can be written in the form $A\mathbf{x} = \mathbf{b}$ as

$$\begin{bmatrix} 1 & 0 & 1 \\ 0 & 1 & -1 \\ -1 & 1 & -1 \end{bmatrix} \begin{bmatrix} x_1 \\ x_2 \\ x_3 \end{bmatrix} = \begin{bmatrix} 2 \\ 1 \\ 0 \end{bmatrix}.$$

Noting that this coefficient matrix is the same as the matrix A in the previous problem, we have that

$$\mathbf{x} = A^{-1}\mathbf{b} = \begin{bmatrix} 0 & 1 & -1 \\ 1 & 0 & 1 \\ 1 & -1 & 1 \end{bmatrix} \begin{bmatrix} 2 \\ 1 \\ 0 \end{bmatrix} = \begin{bmatrix} 1 \\ 2 \\ 1 \end{bmatrix}.$$

4. (30 points) Find the general solution of the following system:

$$\begin{array}{ccccc} x_1 & -2x_2 & +x_3 & & = 0 \\ & x_2 & +x_3 & -x_4 & = 1. \end{array}$$

Answer:

Using an augmented matrix, we have

$$\begin{bmatrix} 1 & -2 & 1 & 0 & 0 \\ 0 & 1 & 1 & -1 & 1 \end{bmatrix} \rightarrow_{2R_2+R_1\rightarrow R_1} \begin{bmatrix} 1 & 0 & 3 & -2 & 2 \\ 0 & 1 & 1 & -1 & 1 \end{bmatrix}.$$

This gives that the solution is

$$\begin{cases} x_1 = 2 + 2x_4 - 3x_3 \\ x_2 = 1 + x_4 - x_3 \\ x_3 \text{ is free} \\ x_4 \text{ is free.} \end{cases}$$

5. (40 points) Let A and B be $n \times n$ matrices. Prove: $(A-B)(A+B) = A^2 - B^2$ if and only if A commutes with B.

Answer:

Proof. Let A and B be as given. We have that

$$(A-B)(A+B) = AA - BA + AB - BB = A^2 - B^2 + AB - BA. \tag{1}$$

If A and B commute, $AB - BA = 0$, and (1) simplifies to $(A-B)(A+B) = A^2 - B^2$. On the other hand, if A and B do not commute, $AB - BA \neq 0$, in which case

$$(A-B)(A+B) - (A^2 - B^2) = AB - BA \neq 0$$

and so $(A-B)(A+B) \neq A^2 - B^2$. □

6. (40 points) TRUE/FALSE (You need not show your work on these problems):

(a) For $n \times n$ matrices A, B, and C,

$$A(B+C) = BA + CA.$$

(b) If $AB = BA$ for the matrices A and B, then

$$A^T B^T = B^T A^T.$$

(c) $E = \begin{bmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 1 & 1 & 0 \end{bmatrix}$ is an elementary matrix.

(d) The mapping

$$\begin{aligned} T: \quad \mathbb{R}^3 &\to \mathbb{R}^3 \\ \begin{bmatrix} x_1 \\ x_2 \\ x_3 \end{bmatrix} &\mapsto \begin{bmatrix} x_1 \\ x_2 \\ -x_3 \end{bmatrix} \end{aligned}$$

is a linear transformation.

(e) If A is the standard matrix for the linear transformation of $\mathbb{R}^2$ which maps an arbitrary vector $\mathbf{x} \mapsto -2\mathbf{x}$, then $A_{12} = 0$.

Answer:

(a) False. $A(B+C) = AB + AC \neq BA + CA$ in general.

(b) True. If $AB = BA$, then $(AB)^T = (BA)^T$, i.e. $B^T A^T = A^T B^T$.

(c) False. To get this matrix from I_3 requires at least 2 row operations, while an elementary matrix is derived from a single operation.

(d) True.

$$\begin{aligned} T(c\mathbf{x} + d\mathbf{y}) = T\left(\begin{bmatrix} cx_1 + dy_1 \\ cx_2 + y_2 \\ cx_3 + dy_3 \end{bmatrix}\right) = \begin{bmatrix} cx_1 + dy_1 \\ cx_2 + y_2 \\ -cx_3 - dy_3 \end{bmatrix} = c\begin{bmatrix} x_1 \\ x_2 \\ -x_3 \end{bmatrix} + d\begin{bmatrix} y_1 \\ y_2 \\ -y_3 \end{bmatrix} \\ = cT(\mathbf{x}) + dT(\mathbf{y}). \end{aligned}$$

(e) True. The standard matrix for this transformation is $\begin{bmatrix} -2 & 0 \\ 0 & -2 \end{bmatrix}$.

7. (0 points) BONUS: Show that the transpose of an elementary matrix is an elementary matrix.

Answer:

Looking at the three types of operations represented by an elementary matrix E, we have:

(a) Multiplying a row by a nonzero constant. Here, the transpose of E is the same as the matrix itself, so it is clearly elementary.

(b) Switching one row with another. Here, the transpose of E is again the same as the matrix itself, so once again, it is elementary.

(c) Adding a nonzero multiple of one row to another. Suppose we add $kR_i + R_j \rightarrow R_j$ for some $k \neq 0$. The resulting elementary matrix E has the same elements as the identity matrix with the lone exception that $E_{ji} = k \neq 0$. Taking the transpose creates a matrix F with the same elements as the identity matrix with the lone exception that $F_{ij} = k$. However, this is the same as the elementary matrix obtained by the operation $kR_j + R_i \rightarrow R_i$, and so $F = E^T$ is elementary as well.