MATH22 - Linear Algebra with Applications

Final Exam ANSWERS

August 28, 2007

1. (25 points) True or False. You need not show your work.

(a) For any integers m and n, $L(\mathbb{R}^n, \mathbb{R}^m)$ is a finite dimensional vector space.

(b) If A is an $n \times n$ matrix, dim Col A + dim Row $A^T = n$.

(c) If $A = A^T$, every eigenvalue of A has multiplicity ≥ 2.

(d) A real, square matrix is orthogonal if and only if its rows are mutually orthogonal, normal vectors.

(e) Every square matrix is similar to its transpose.

Answer:

(a) True, as we saw in the bonus problem for Exam II, where we showed it has dimension mn.

(b) False. If A is invertible, each part of the left hand side of the equation is equal to n, for a sum of $2n$.

(c) False. Try $A = [1]$.

(d) True, as shown in class.

(e) True, since the transpose has the same characteristic equation.

2. (45 points) Let $A = \begin{bmatrix} 1 & 2 & 0 & 3 \\ -1 & 1 & -3 & -3 \\ 0 & 2 & -2 & 0 \\ 2 & 4 & 0 & 6 \end{bmatrix}$.

(a) Find a basis for Nul A.

(b) Use the result of part (a) to write $\mathbf{v} = \begin{bmatrix} 6 \\ 12 \\ -6 \\ 12 \end{bmatrix}$ as $\mathbf{v}_1 + \mathbf{v}_2$ where $\mathbf{v}_1 \in \text{Nul } A$ and $\mathbf{v}_2 \in \text{Row } A$.

Answer:

(a) Setting up the augmented matrix for the homogeneous system,

$$\begin{bmatrix} 1 & 2 & 0 & 3 & 0 \\ -1 & 1 & -3 & -3 & 0 \\ 0 & 2 & -2 & 0 & 0 \\ 2 & 4 & 0 & 6 & 0 \end{bmatrix} \xrightarrow[-2R_1+R_4 \to R_4]{R_1+R_2 \to R_2} \begin{bmatrix} 1 & 2 & 0 & 3 & 0 \\ 0 & 3 & -3 & 0 & 0 \\ 0 & 2 & -2 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 \end{bmatrix} \xrightarrow{\frac{1}{3}R_2 \to R_2}$$

$$\begin{bmatrix} 1 & 2 & 0 & 3 & 0 \\ 0 & 1 & -1 & 0 & 0 \\ 0 & 2 & -2 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 \end{bmatrix} \xrightarrow{-R_2+R_3 \to R_3} \begin{bmatrix} 1 & 2 & 0 & 3 & 0 \\ 0 & 1 & -1 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 \end{bmatrix} \xrightarrow{-2R_2+R_1 \to R_1}$$

$$\begin{bmatrix} 1 & 0 & 2 & 3 & 0 \\ 0 & 1 & -1 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 \end{bmatrix},$$

so the general solution to the homogeneous equation is

$$\begin{cases} x_1 & = -2x_3 - 3x_4 \\ x_2 & = x_3 \\ x_3, x_4 & \text{are free.} \end{cases}$$

Thus, a basis for Nul A is

$$\left\{ \begin{bmatrix} -2 \\ 1 \\ 1 \\ 0 \end{bmatrix}, \begin{bmatrix} -3 \\ 0 \\ 0 \\ 1 \end{bmatrix} \right\}.$$

(b) To do orthogonal projection onto Nul A, we need an orthogonal basis for Nul A. We may take $\mathbf{x}_1 = \begin{bmatrix} -2 \\ 1 \\ 1 \\ 0 \end{bmatrix}$, and put

$$\mathbf{x}_2 = \begin{bmatrix} -3 \\ 0 \\ 0 \\ 1 \end{bmatrix} - \frac{\begin{bmatrix} -3 \\ 0 \\ 0 \\ 1 \end{bmatrix} \cdot \begin{bmatrix} -2 \\ 1 \\ 1 \\ 0 \end{bmatrix}}{\begin{bmatrix} -2 \\ 1 \\ 1 \\ 0 \end{bmatrix} \cdot \begin{bmatrix} -2 \\ 1 \\ 1 \\ 0 \end{bmatrix}} \begin{bmatrix} -2 \\ 1 \\ 1 \\ 0 \end{bmatrix} = \begin{bmatrix} -1 \\ -1 \\ -1 \\ 1 \end{bmatrix}.$$

Then, since Row $A = (\text{Nul } A)^{\perp}$,

$$\begin{aligned} \mathbf{v}_1 = \text{proj}_{\text{Nul } A}\mathbf{v} &= \frac{\begin{bmatrix} -2 \\ 1 \\ 1 \\ 0 \end{bmatrix} \cdot \begin{bmatrix} 6 \\ 12 \\ -6 \\ 12 \end{bmatrix}}{\begin{bmatrix} -2 \\ 1 \\ 1 \\ 0 \end{bmatrix} \cdot \begin{bmatrix} -2 \\ 1 \\ 1 \\ 0 \end{bmatrix}} \begin{bmatrix} -2 \\ 1 \\ 1 \\ 0 \end{bmatrix} + \frac{\begin{bmatrix} -1 \\ -1 \\ -1 \\ 1 \end{bmatrix} \cdot \begin{bmatrix} 6 \\ 12 \\ -6 \\ 12 \end{bmatrix}}{\begin{bmatrix} -1 \\ -1 \\ -1 \\ 1 \end{bmatrix} \cdot \begin{bmatrix} -1 \\ -1 \\ -1 \\ 1 \end{bmatrix}} \cdot \begin{bmatrix} -1 \\ -1 \\ -1 \\ 1 \end{bmatrix} \\ &= \frac{-6}{6} \begin{bmatrix} -2 \\ 1 \\ 1 \\ 0 \end{bmatrix} + \frac{0}{4} \begin{bmatrix} -1 \\ -1 \\ -1 \\ 1 \end{bmatrix} \\ &= \begin{bmatrix} 2 \\ -1 \\ -1 \\ 0 \end{bmatrix}. \end{aligned}$$

Then,

$$\mathbf{v}_2 = \mathbf{v} - \mathbf{v}_1 = \begin{bmatrix} 6 \\ 12 \\ -6 \\ 12 \end{bmatrix} - \begin{bmatrix} 2 \\ -1 \\ -1 \\ 0 \end{bmatrix} = \begin{bmatrix} 4 \\ 13 \\ -5 \\ 12 \end{bmatrix}.$$

3. (30 points) Suppose there are three candidates in each presidential election: a Democrat (D), Independent (I), and a Republican (R). An incumbent representative has a 50% chance of re-election. A Democratic challenger has a 40% chance of unseating an independent incumbent and a 30% chance of winning election against a Republican incumbent. An Independent has a 20% chance of being elected over a Democratic incumbent.

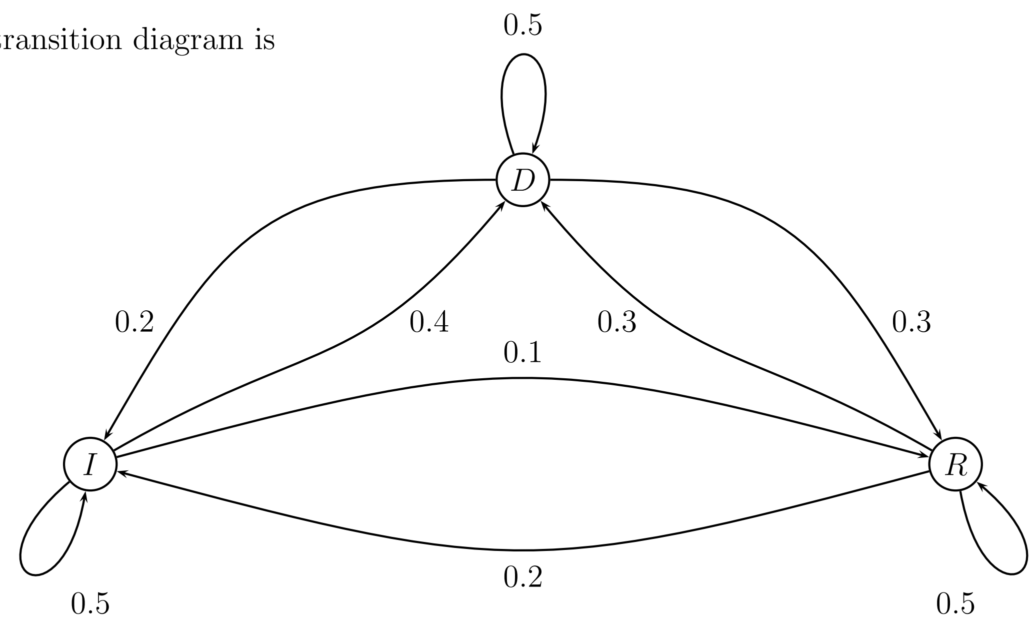
(a) Draw a transition diagram to represent this situation.

(b) Give a transition matrix that represents this situation.

(c) If the current president is a Republican, what are the probabilities that it will have a Democrat, Independent, or Republican in the next election? Two elections from now?

Answer:

(a) The transition diagram is



(b) The transition matrix is

$$P = \begin{array}{c} D \\ I \\ R \end{array} \overset{\begin{array}{ccc} D & I & R \end{array}}{\begin{bmatrix} .5 & .4 & .3 \\ .2 & .5 & .2 \\ .3 & .1 & .5 \end{bmatrix}}.$$

(c) We have

$$S_0 = \begin{bmatrix} 0 \\ 0 \\ 1 \end{bmatrix} \begin{array}{c} D \\ I \\ R \end{array},$$

so

$$S_1 = PS_0 = \begin{bmatrix} .5 & .4 & .3 \\ .2 & .5 & .2 \\ .3 & .1 & .5 \end{bmatrix} \begin{bmatrix} 0 \\ 0 \\ 1 \end{bmatrix} = \begin{bmatrix} .3 \\ .2 \\ .5 \end{bmatrix} \begin{matrix} D \\ I \\ R \end{matrix}$$

and

$$S_2 = PS_1 = \begin{bmatrix} .5 & .4 & .3 \\ .2 & .5 & .2 \\ .3 & .1 & .5 \end{bmatrix} \begin{bmatrix} .3 \\ .2 \\ .5 \end{bmatrix} = \begin{bmatrix} .38 \\ .26 \\ .36 \end{bmatrix} \begin{matrix} D \\ I \\ R \end{matrix} \quad .$$

4. (20 points) Show that the set of symmetric $n \times n$ matrices is a subspace of $M_{n\times n}$. (Recall A is symmetric if $A = A^T$.)

Answer:

Let $S = \{A \in M_{n\times n} : A = A^T\}$. Since $0 = 0^T$, $0 \in S$. If $A, B \in S$, then

$$(A + B)^T = A^T + B^T = A + B,$$

so $A + B \in S$. If $A \in S$ and $c \in \mathbb{R}$,

$$(cA)^T = cA^T = cA,$$

so $cA \in S$. Thus, S is a subspace of $M_{n\times n}$.

5. (30 points) Prove $(a_1^2 + a_2^2 + \cdots + a_n^2)^2 \leq n(a_1^4 + a_2^4 + \cdots + a_n^4)$ for any integer n and any $a_1, a_2, \ldots, a_n \in \mathbb{R}$. (Hint: Use Cauchy-Schwarz.)

Answer:

Let $\mathbf{u}, \mathbf{v} \in \mathbb{R}^n$, with $\mathbf{u} = \begin{bmatrix} 1 \\ 1 \\ \vdots \\ 1 \end{bmatrix}$ and $\mathbf{v} = \begin{bmatrix} a_1^2 \\ a_2^2 \\ \vdots \\ a_n^2 \end{bmatrix}$. Then, by Cauchy-Schwarz,

$$(a_1^2 + a_2^2 + \cdots + a_n^2) = \mathbf{u} \cdot \mathbf{v} \leq ||\mathbf{u}||\,||\mathbf{v}|| = \sqrt{n}\sqrt{a_1^4 + a_2^4 + \cdots + a_n^4}.$$

Squaring both sides gives the result.

6. (25 points) Let $S : \mathbb{P}_4 \to \mathbb{P}_5$ be the linear operator for which $S(f) = \int_0^t f(x)\,dx$. Find the matrix representation for S with respect to the standard bases for $\mathbb{P}_4$ and $\mathbb{P}_5$.

Answer:

Since $S(t^n) = \frac{t^{n+1}}{n}$ and since the standard bases in question are $\beta_4 = \{1, t, t^2, t^3, t^4\}$ and $\beta_5 = \{1, t, t^2, t^3, t^4, t^5\}$, the standard matrix for S is:

$$\left[\ [S(1)]_{\beta_5} \quad [S(t)]_{\beta_5} \quad [S(t^2)]_{\beta_5} \quad [S(t^3)]_{\beta_5} \quad [S(t^4)]_{\beta_5}\ \right] = \begin{bmatrix} 0 & 0 & 0 & 0 & 0 \\ 1 & 0 & 0 & 0 & 0 \\ 0 & \frac{1}{2} & 0 & 0 & 0 \\ 0 & 0 & \frac{1}{3} & 0 & 0 \\ 0 & 0 & 0 & \frac{1}{4} & 0 \\ 0 & 0 & 0 & 0 & \frac{1}{5} \end{bmatrix}.$$

7. (25 points) Find the general solution for the equation $\begin{bmatrix} 0 & 1 & -4 \\ 5 & 4 & 9 \\ 2 & 2 & 2 \end{bmatrix} \mathbf{x} = \begin{bmatrix} 0 \\ 5 \\ 2 \end{bmatrix}$.

Answer:

Setting up the augmented matrix, we see

$$\begin{bmatrix} 0 & 1 & -4 & 0 \\ 5 & 4 & 9 & 5 \\ 2 & 2 & 2 & 2 \end{bmatrix} \rightarrow_{R_1 \leftrightarrow R_3} \begin{bmatrix} 2 & 2 & 2 & 2 \\ 5 & 4 & 9 & 5 \\ 0 & 1 & -4 & 0 \end{bmatrix} \rightarrow_{\frac{1}{2}R_1 \rightarrow R_1} \begin{bmatrix} 1 & 1 & 1 & 1 \\ 5 & 4 & 9 & 5 \\ 0 & 1 & -4 & 0 \end{bmatrix} \rightarrow_{-5R_1+R_2 \rightarrow R_2}$$

$$\begin{bmatrix} 1 & 1 & 1 & 1 \\ 0 & -1 & 4 & 0 \\ 0 & 1 & -4 & 0 \end{bmatrix} \rightarrow_{-R_2 \rightarrow R_2} \begin{bmatrix} 1 & 1 & 1 & 1 \\ 0 & 1 & -4 & 0 \\ 0 & 1 & -4 & 0 \end{bmatrix} \rightarrow_{-R_2+R_3 \rightarrow R_3} \begin{bmatrix} 1 & 1 & 1 & 1 \\ 0 & 1 & -4 & 0 \\ 0 & 0 & 0 & 0 \end{bmatrix} \rightarrow_{-R_2+R_1 \rightarrow R_1}$$

$$\begin{bmatrix} 1 & 0 & 5 & 1 \\ 0 & 1 & -4 & 0 \\ 0 & 0 & 0 & 0 \end{bmatrix},$$

so the general solution is

$$\begin{cases} x_1 & = 1 - 5x_3 \\ x_2 & = 4x_3 \\ x_3 & \text{is free.} \end{cases}$$

Equivalently, we may write the general solution as

$$\mathbf{x} = \begin{bmatrix} 1 \\ 0 \\ 0 \end{bmatrix} + t \begin{bmatrix} -5 \\ 4 \\ 1 \end{bmatrix},$$

where t varies over the real numbers.

8. (40 points)

(a) Find all eigenvalues and eigenvectors of $A = \begin{bmatrix} 1 & 1 & -1 \\ 1 & -2 & 0 \\ 0 & -3 & 1 \end{bmatrix}$.

(b) Use part (a) to write $A = PDP^{-1}$ for P a 3×3 invertible matrix and D a 3×3 diagnonal matrix. (You need not compute P^{-1}.)

Answer:

(a) We have

$$\begin{aligned}\det(\lambda I - A) &= \begin{vmatrix} \lambda-1 & -1 & 1 \\ -1 & \lambda+2 & 0 \\ 0 & 3 & \lambda-1 \end{vmatrix} \\ &= (\lambda-1)(\lambda+2)(\lambda-1) - 3 - (\lambda-1) \\ &= (\lambda^2 - 2\lambda + 1)(\lambda+2) - 3 - \lambda + 1 \\ &= \lambda^3 - 2\lambda^2 + \lambda + 2\lambda^2 - 4\lambda + 2 - 3 - \lambda + 1 \\ &= \lambda^3 - 4\lambda,\end{aligned}$$

so the eigenvalues are $\lambda = 0, \pm 2$.

For $\lambda = 0$:We can row reduce to see that

$$\begin{bmatrix} 1 & 1 & -1 & 0 \\ 1 & -2 & 0 & 0 \\ 0 & -3 & 1 & 0 \end{bmatrix} \sim \begin{bmatrix} 1 & 0 & -\frac{2}{3} & 0 \\ 0 & 1 & -\frac{1}{3} & 0 \\ 0 & 0 & 0 & 0 \end{bmatrix},$$

so we may choose the eigenvector

$$\mathbf{v}_0 = \begin{bmatrix} 2 \\ 1 \\ 3 \end{bmatrix}.$$

For $\lambda = 2$:We can row reduce to see that

$$\begin{bmatrix} 1 & -1 & 1 & 0 \\ -1 & 4 & 0 & 0 \\ 0 & 3 & 1 & 0 \end{bmatrix} \sim \begin{bmatrix} 1 & 0 & \frac{4}{3} & 0 \\ 0 & 1 & \frac{1}{3} & 0 \\ 0 & 0 & 0 & 0 \end{bmatrix},$$

so we may choose the eigenvector

$$\mathbf{v}_2 = \begin{bmatrix} -4 \\ -1 \\ 3 \end{bmatrix}.$$

For $\lambda = -2$:We can row reduce to see that

$$\begin{bmatrix} -3 & -1 & 1 & 0 \\ -1 & 0 & 0 & 0 \\ 0 & 3 & -3 & 0 \end{bmatrix} \sim \begin{bmatrix} 1 & 0 & 0 & 0 \\ 0 & 1 & -1 & 0 \\ 0 & 0 & 0 & 0 \end{bmatrix},$$

so we may choose the eigenvector

$$\mathbf{v}_{-2} = \begin{bmatrix} 0 \\ -1 \\ 1 \end{bmatrix}.$$

(b) From part (a), we may put

$$P = \begin{bmatrix} \mathbf{v}_{-2} & \mathbf{v}_0 & \mathbf{v}_2 \end{bmatrix}$$

and

$$D = \begin{bmatrix} -2 & 0 & 0 \\ 0 & 0 & 0 \\ 0 & 0 & 2 \end{bmatrix}.$$

Thus,

$$A = PDP^{-1} = \begin{bmatrix} 0 & 2 & -4 \\ 1 & 1 & -1 \\ 1 & 3 & 3 \end{bmatrix} \begin{bmatrix} -2 & 0 & 0 \\ 0 & 0 & 0 \\ 0 & 0 & 2 \end{bmatrix} \begin{bmatrix} 0 & 2 & -4 \\ 1 & 1 & -1 \\ 1 & 3 & 3 \end{bmatrix}^{-1}.$$

9. (0 points) BONUS: Name and describe two of the applications of linear algebra we saw in this course.

Answer:

Any of the topics covered are acceptable, including, among others,

(a) covariance-based facial recognition

(b) Google's PageRank algorithm

(c) network flow (traffic, electric current, etc.)

(d) balancing chemical equations

(e) cryptography

(f) linear programming

(g) Leontief Input-Output analysis

(h) magic squares

(i) computer graphics

(j) modeling a stochastic system (card trick, migration, etc.)