SOLUTIONS

Math 22: Linear Algebra. MIDTERM 2

Barnett 8/8/06

(revised).

2 hrs, no calculators. Please answer all six questions. Answer on this sheet. Your NAME:

1. [11 points]

   (a) Compute (without using row swaps) the LU decomposition of

$$A = \begin{bmatrix} 2 & 1 & 1 & 0 \\ -4 & -3 & 2 & 0 \\ 6 & 2 & 0 & 1 \end{bmatrix}$$

Row reduce A to EF:

$$A \sim \begin{bmatrix} 2 & 1 & 1 & 0 \\ 0 & -1 & 4 & 0 \\ 0 & -1 & -3 & 1 \end{bmatrix} \quad \text{gives} \quad L = \begin{bmatrix} 1 & & \\ -2 & 1 & \\ 3 & \cdot & 1 \end{bmatrix}$$

ones on diag since no rescalings

1st col. only can be filled

$$\sim \begin{bmatrix} 2 & 1 & 1 & 0 \\ 0 & -1 & 4 & 0 \\ 0 & 0 & -7 & 1 \end{bmatrix} = U \quad \text{gives} \quad L = \begin{bmatrix} 1 & & \\ -2 & 1 & \\ 3 & 1 & 1 \end{bmatrix}$$

$R_3 \leftarrow R_3 - R_2$ so +1 entered in 3,2 entry of L.

signs of L were important

   (b) Counting from the left as usual, which is the *first* column of A that can be written as linear combination of the previous ones, and why?

A's R.E.F. has structure $\begin{bmatrix} \blacksquare & * & * & * \\ & \blacksquare & * & * \\ & & \blacksquare & * \end{bmatrix}$

3 pivots: $x_1\ x_2\ x_3$. free variable is x_4

1st 3 columns are the identity.

write $A = [\vec{a}_1 \vec{a}_2 \vec{a}_3 \vec{a}_4]$

Row reduction preserves the linear dependence relation between columns,

so $\vec{a}_2$ is not a multiple of $\vec{a}_1$, neither is $\vec{a}_3$ a lin. comb. of $\vec{a}_1, \vec{a}_2$ } by looking at corresponding columns of R.E.F (they are 3×3 identity)

But, $\vec{a}_4$ is in the span of $\vec{a}_1, \vec{a}_2, \vec{a}_3$, so is the first such column.

Note it's always the <u>first free variable column</u>.

(c) Let B be any lower triangular matrix with non-zero entries on the diagonal. Prove that the inverse of B exists and is also lower triangular. [Hint: elementary row operations].

B is invertible since its determinant is the product of diagonal entries, $\Rightarrow$ nonzero.

Any such matrix B can be reached by applying elementary row operations to I, ie

$$B = E_p \cdots E_1 I.$$

Furthermore each such row op corresponds to adding a multiple of a row to a lower row, ie $E_j = \begin{bmatrix} 1 & & \\ & 1 & \\ -2 & & 1 \end{bmatrix}$ but not $\begin{bmatrix} 1 & & -2 \\ & 1 & \\ & & 1 \end{bmatrix}$

so $$\underbrace{(E_p \cdots E_1)^{-1}}_{= E_1^{-1} E_2^{-1} \cdots E_p^{-1}} B = I$$

by property of inverses $(AB)^{-1} = B^{-1}A^{-1}$.

each of these exists, since row ops. invertible.
And each is also a lower triangular elementary matrix
E.g. $\begin{bmatrix} 1 & & \\ & 1 & \\ 2 & & 1 \end{bmatrix}$ is the inverse of $\begin{bmatrix} 1 & & \\ & 1 & \\ -2 & & 1 \end{bmatrix}$.

So $B^{-1} = E_1^{-1} \cdots E_p^{-1}$ exist and is lower triangular.

2. [12 points]

(a) Find the real eigenvalues (if they exist) and multiplicities of $\begin{bmatrix} -2 & 1 \\ 0 & -2 \end{bmatrix}$

$$\det \begin{bmatrix} -2-\lambda & 1 \\ 0 & -2-\lambda \end{bmatrix} = (-2-\lambda)^2 - 1(0) = (-2-\lambda)^2 = 0$$

so $\lambda = -2$, multiplicity 2.

(b) Find the real eigenvalues (if they exist) and multiplicities of $\begin{bmatrix} 2 & 1 \\ -1 & 2 \end{bmatrix}$

$$\det \begin{bmatrix} 2-\lambda & 1 \\ -1 & 2-\lambda \end{bmatrix} = (2-\lambda)(2-\lambda) - (-1)1 = 4 - 4\lambda + \lambda^2 + 1$$

$$= \lambda^2 - 4\lambda + 5. \qquad \lambda = \frac{1}{2}\left[+4 \pm \sqrt{\underbrace{4^2 - 4(5)}_{<0}}\right.$$

negative square-root $\Rightarrow$ complex, no real roots.

(c) Find the eigenvalues (which are all real) and multiplicities of

$$A = \begin{bmatrix} 1 & 0 & 0 \\ 2 & 3 & 1 \\ -1 & 0 & 1 \end{bmatrix}. \qquad \det(A - \lambda I) = \begin{vmatrix} 1-\lambda & 0 & 0 \\ 2 & 3-\lambda & 1 \\ -1 & 0 & 1-\lambda \end{vmatrix}$$

notice only one nonzero entry $\Rightarrow$ use cofactor.

$$= (1-\lambda)\left[(3-\lambda)(1-\lambda) - 1(0)\right] = (1-\lambda)^2(3-\lambda) = 0$$

$\lambda = 1$ (multiplicity 2), $\lambda = 3$.

(d) Find a basis for the eigenspace associated with the above <u>double eigenvalue</u>. What is its dimension?

i.e. the $\lambda = 1$ one

use $\lambda=1$

$$A - \lambda I = \begin{bmatrix} 0 & 0 & 0 \\ 2 & 2 & 1 \\ -1 & 0 & 0 \end{bmatrix} \underset{\text{swaps}}{\sim} \begin{bmatrix} -1 & 0 & 0 \\ 2 & 2 & 1 \\ 0 & 0 & 0 \end{bmatrix} \sim \begin{bmatrix} 1 & 0 & 0 \\ 0 & 2 & 1 \\ 0 & 0 & 0 \end{bmatrix}$$

$$\underset{\text{rescale } R_2}{\sim} \begin{bmatrix} 1 & 0 & 0 \\ 0 & 1 & 1/2 \\ 0 & 0 & 0 \end{bmatrix}$$ now in R.E.F.

x_3 free. $\therefore$ $\dim \operatorname{Nul}(A - \lambda I) = 1$ (surprise since λ was double eigenvalue!)

$$\Rightarrow \quad \begin{aligned} x_1 &= 0 \\ x_2 &= -\tfrac{1}{2}x_3 \\ x_3 &= x_3. \end{aligned} \qquad \text{so} \quad \vec{x} = \begin{bmatrix} 0 \\ -1/2 \\ 1 \end{bmatrix}$$

or any nonzero multiple, is the only eigenvector.

The eigenspace has dimension 1.

for this eigenspace: geometric mult (1) < algebraic mult (2).

3. [10 points]

(a) True/false: Two eigenvectors with the same $\boxed{\text{eigenvalue}}$ (call it λ) are always linearly independent? A eigenvector is any nonzero $\vec{x}$ such that $A\vec{x} = \lambda\vec{x}$

So $\vec{x}$ & $2\vec{x}$ are <u>both</u> eigenvectors, and certainly are <u>not</u> L.I.

Note: eigenvectors in the same eigenspace can be L.I. if $\dim \mathrm{Nul}(A-\lambda I) > 1$.

(b) What is the rank of a 5×3 matrix if a <u>basis for its null space contains only one vector</u>?

means, one free var. e.g. $\begin{bmatrix} \blacksquare & * & * \\ & & * \\ & & * \\ & & \blacksquare \\ & & \end{bmatrix}$

$\Rightarrow$ only 2 pivots, <u>rank=2</u>.

or recall theorem, $\mathrm{rank}\,A + \dim \mathrm{Nul}\,A = n$.

(c) True/false: Given a $n \times n$ matrix A, if $A\mathbf{x} = \mathbf{b}$ is inconsistent for some $\mathbf{b}$ then A must have at least one real eigenvalue?

A must <u>not</u> have complete set of n pivots if can be inconsistent.
So A is not invertible. So zero is an eigenvalue, and is certainly real.

(d) True/false: The set $W = \left\{ \begin{bmatrix} x \\ y \end{bmatrix} : x + 2y = 0 \right\}$ is a subspace of $\mathbb{R}^2$?

The set is the nullspace of the 1×2 matrix $[1 \; 2]$, so this proves it is a subspace. Its elements are in $\mathbb{R}^2$, since they have 2 components.

(e) Explain why a $n \times n$ matrix can have at most n eigenvalues.

Characteristic polynomial is $\det(A - \lambda I) = (-1)^n \lambda^n + c_{n-1}\lambda^{n-1} + \cdots c_1\lambda + c_0$

an nth-degree polynomial.

Such a polynomial can have at most n real roots, which are the eigenvalues. by the Fundamental Thm. of Algebra.

4. [7 points] Compute the determinants of the following matrices: [Hint: in each case one method is much easier than the other]

(a) $\begin{bmatrix} 2 & 0 & 6 \\ 0 & 7 & 0 \\ 1 & 0 & 3 \end{bmatrix}$

Lots of zeros $\Rightarrow$ Cofactor expansion.

$$\det A = 7 \cdot \underbrace{\begin{vmatrix} 2 & 6 \\ 1 & 3 \end{vmatrix}}_{ad-bc \,=\, 2(3)-6(1) \,=\, 0.}$$

$$= 7 \cdot 0 = 0.$$

(b) $\begin{bmatrix} 1 & 3 & 2 \\ 2 & 6 & 9 \\ 3 & 7 & 29 \end{bmatrix}$

dense (no zeros) and first 2 rows look similar $\Rightarrow$ row reduce.

$$\begin{vmatrix} 1 & 3 & 2 \\ 2 & 6 & 9 \\ 3 & 7 & 29 \end{vmatrix} \begin{matrix} \\ R_2-2R_1 \\ R_3-3R_1 \end{matrix} = \begin{vmatrix} 1 & 3 & 2 \\ 0 & 0 & 5 \\ 0 & -2 & 23 \end{vmatrix} = - \underbrace{\begin{vmatrix} 1 & 3 & 2 \\ 0 & -2 & 23 \\ 0 & 0 & 5 \end{vmatrix}}_{\text{upper triangular so det = product of diagonal entries.}} = -(1)(-2)(5) = \underline{+10}.$$

since odd # of swaps.

5. [11 points]

5 [redacted] The matrix A has been converted to reduced echelon form as follows

$$A = \begin{bmatrix} -2 & -4 & 1 & 0 & 4 \\ 0 & 0 & 0 & 3 & 3 \\ 1 & 2 & 2 & 1 & 5 \\ 3 & 6 & 0 & -2 & -2 \end{bmatrix} \sim \begin{bmatrix} 1 & 2 & 0 & 0 & -1 \\ 0 & 0 & 1 & 0 & 2 \\ 0 & 0 & 0 & 1 & 1 \\ 0 & 0 & 0 & 0 & 0 \end{bmatrix}.$$

$\vec{a}_1 \quad \vec{a}_3 \quad \vec{a}_4$

Crucial: categorize.
pivot cols: 1, 3, 4.
free vars: 2, 5

(a) Write down a basis for the column space of A:

basis for Col A = $\{\vec{a}_1, \vec{a}_3, \vec{a}_4\} = \left\{ \begin{bmatrix} -2 \\ 0 \\ 1 \\ 3 \end{bmatrix}, \begin{bmatrix} 1 \\ 0 \\ 2 \\ 0 \end{bmatrix}, \begin{bmatrix} 0 \\ 3 \\ 1 \\ -2 \end{bmatrix} \right\}$

(b) Write down a basis for the null space of A:

write parametric form for solution set to $A\vec{x} = \vec{0}$: $\quad x_2, x_5$ free.

$$\left.\begin{aligned} x_1 &= -2x_2 + x_5 \\ x_2 &= x_2 \\ x_3 &= -2x_5 \\ x_4 &= -x_5 \\ x_5 &= x_5 \end{aligned}\right\} \text{vector eqn:} \quad \vec{x} = x_2 \begin{bmatrix} -2 \\ 1 \\ 0 \\ 0 \\ 0 \end{bmatrix} + x_5 \begin{bmatrix} 1 \\ 0 \\ -2 \\ -1 \\ 1 \end{bmatrix}$$

these vectors are the desired basis.

(c) What is the dimension of the subspace consisting of all possible vectors b such that $A\mathbf{x} = \mathbf{b}$ for some $\mathbf{x}$?

This defines the column space; so $\dim \operatorname{Col} A = \operatorname{rank} A = \#\text{pivots} = 3$

(d) What is the dimension of the subspace consisting of all solutions to the equation $A\mathbf{x} = 0$? ← defines the nullspace of A.

$\dim \operatorname{Nul} A = \#$ free vars $= 2$

(e) Explain why the first 3 rows of the R.E.F. of A form a basis for Row A.

This relies on 3 facts: i) Span (rows of A) = Span (rows of R.E.F.) since elementary row ops. are reversible then the spaces must be the same (see book. p. 263).

ii) The last row $[0\ 0\ 0\ 0\ 0]$ can be dropped from the list in the R.E.F. without affecting their span. (It has no effect!).

iii) The first 3 rows of R.E.F. are L.I. since, counting from 3rd one backwards, $\vec{r}_2$ is not multiple of $\vec{r}_3$, and $\vec{r}_1$ not in span$\{\vec{r}_2, \vec{r}_3\}$ since the pivots lie above zeros in the lower rows.

Spans the subspace
+
Lin. Indep.
⇒ Basis.

6. [9 points]

(a) Does the set $\{1+t^2, t+t^2, t-t^2\}$ form a basis for the vector space of all polynomials of the form $a+bt+ct^2$? Explain what criteria you tested, and if each test failed or passed.

commonly known as $\mathbb{P}_2$.

Since coordinate map $\mathbb{P}_2 \to \mathbb{R}^3$ is isomorphism, we may work in $\mathbb{R}^3$ instead.

The coordinates of the set are $\left\{ \begin{bmatrix} 1 \\ 0 \\ 1 \end{bmatrix}, \begin{bmatrix} 0 \\ 1 \\ 1 \end{bmatrix}, \begin{bmatrix} 0 \\ 1 \\ -1 \end{bmatrix} \right\} \xrightarrow[\text{matrix}]{\text{stack as}} A = \begin{bmatrix} 1 & 0 & 0 \\ 0 & 1 & 1 \\ 1 & 1 & -1 \end{bmatrix}$

If rank $A = 3$ then the set spans $\mathbb{R}^3$ and is L.I, so is a basis.

$$A \sim \begin{bmatrix} 1 & 0 & 0 \\ 0 & 1 & 1 \\ 0 & 1 & -1 \end{bmatrix} \sim \begin{bmatrix} 1 & 0 & 0 \\ 0 & 1 & 1 \\ 0 & 0 & -2 \end{bmatrix}$$

$\leftarrow$ 3 pivots, full rank $\Rightarrow$ Yes, both tests passed.

(b) The set of vectors $\left\{ \begin{bmatrix} 1 \\ 0 \\ 0 \end{bmatrix}, \begin{bmatrix} 1 \\ 1 \\ 0 \end{bmatrix}, \begin{bmatrix} 1 \\ 1 \\ 1 \end{bmatrix} \right\}$ form a basis $\mathcal{B}$ for $\mathbb{R}^3$. If $\mathbf{x} = \begin{bmatrix} 1 \\ 2 \\ 3 \end{bmatrix}$, find $[\mathbf{x}]_{\mathcal{B}}$.

$\vec{b}_1 \quad \vec{b}_2 \quad \vec{b}_3$

$[\vec{x}]_B = \begin{pmatrix} c_1 \\ c_2 \\ c_3 \end{pmatrix}$ such that $c_1\vec{b}_1 + c_2\vec{b}_2 + c_3\vec{b}_3 = \vec{x}$

Solve the linear system:

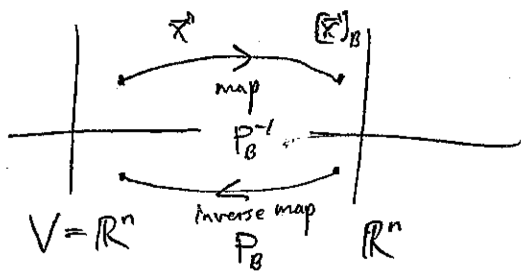
$$\left[\begin{array}{ccc|c} 1 & 1 & 1 & 1 \\ 0 & 1 & 1 & 2 \\ 0 & 0 & 1 & 3 \end{array}\right] \sim \left[\begin{array}{ccc|c} 1 & 1 & 0 & -2 \\ 0 & 1 & 0 & -1 \\ 0 & 0 & 1 & 3 \end{array}\right] \sim \left[\begin{array}{ccc|c} 1 & 0 & 0 & -1 \\ & 1 & 0 & -1 \\ & & 1 & 3 \end{array}\right]$$

already in E.F. — row reduce to REF. — identity $\Rightarrow$ done.

$$[\vec{x}]_B = \begin{bmatrix} -1 \\ -1 \\ 3 \end{bmatrix}$$

(c) Prove that for any basis for $\mathbb{R}^n$ the coordinate mapping $\mathbf{x} \to [\mathbf{x}]_{\mathcal{B}}$ is one-to-one.

note $\vec{x}$ is the RHS "$\vec{b}$"; $[\vec{x}]_B$ the unknown



map is given by the solution to $P_B[\vec{x}]_B = \vec{x}$, where P_B is the change-of-coords matrix given by stacking the basis vectors as columns.

One-to-one means each $[\vec{x}]_B$ can come from only one $\vec{x}$.

But this must hold, since the inverse map $[\vec{x}]_B \to \vec{x}$ is a transformation given by multiplying by P_B, so each $[\vec{x}]_B$ uniquely defines an $\vec{x}$ simply by $\vec{x} = P_B[\vec{x}]_B$. Think about it!

Note: no mention of L.I. of the basis is needed! The situation is backwards compared to usual since $\longrightarrow$ is given by multiplying by P_B^{-1}, not P_B.