Math 22: Linear Algebra. PRACTISE FINAL

No calculators, one letter sheet of notes allowed. Your NAME:

This exam is maybe biased towards later material, to help you practise. I recommend you also go over earlier questions too from the Mid1, Mid2 practise lists.

1. [12 points]A set of vectors in $\mathbb{R}^3$ is given by $W = \left\{ \begin{bmatrix} a+2b+2c \\ -2b+c \\ a+3c \end{bmatrix} \; a, b, c \text{ real} \right\}$.

   (a) Determine whether W is a subspace of $\mathbb{R}^3$. (Explain any tests or results you make use of).

   (b) Find a basis for W. [Hint: recognise W as one fundamental space of a matrix].

(c) Give a definition of the set $W^\perp$.

(d) Find a basis for $W^\perp$. [Hint: use previous hint].

2. [10 points]

(a) Explain how the solution set to $A\mathbf{x} = \mathbf{b}$ relates to the solution set of $A\mathbf{x} = \mathbf{0}$?

(b) Is $B = \left\{ \begin{bmatrix} 1 \\ 1 \\ 0 \end{bmatrix}, \begin{bmatrix} 1 \\ 0 \\ 2 \end{bmatrix} \right\}$ a basis for $W = \text{Span} \left\{ \begin{bmatrix} 0 \\ 1 \\ -2 \end{bmatrix}, \begin{bmatrix} 3 \\ 2 \\ 2 \end{bmatrix} \right\}$? (Why—pretend I always asked why!)

(c) Is $B = \left\{ \begin{bmatrix} 2 \\ -1 \end{bmatrix}, \begin{bmatrix} -4 \\ 2 \end{bmatrix} \right\}$ a basis for $W = \left\{ \begin{bmatrix} x \\ y \end{bmatrix} : x + 2y = 0 \right\}$?

3. [8 points]

(a) If A is a 6-by-8 matrix, and dim Nul $A = 3$, what is rank A ?

(b) Can a system of 3 linear equations in 5 unknowns be unique? If not, is the solution set always a subspace?

(c) V is a finite dimensional vector space. Let $\mathbf{v}_1, \mathbf{v}_2, \ldots, \mathbf{v}_n$ be vectors in V. Give the definition of linear independence of this set of vectors.

4. [12 points]

A given member of the population either votes Democratic (D) or Republican (R). In a given election if they voted D they will randomly vote R 50% of the time, and D 50% of the time, in the next election. However if they voted R they will *always* vote D in the next election.

(a) Write down the stochastic matrix.

(b) If in a given election the voter proproportions are 50% and 50%, find their proportions in the following election.

(c) Find the steady-state probability vector.

(d) Explain why in the long-time limit the proportions must tend to this steady-state vector.

5. [12 points] Categorize the stability of the origin, for the following dynamical systems $\mathbf{x}_{k+1} = A\mathbf{x}_k$.

(a) $A = \begin{bmatrix} 2 & 3 \\ 0 & 2 \end{bmatrix}$

(b) $A = \begin{bmatrix} 1 & 5 \\ 5 & 1 \end{bmatrix}$

(c) $A = \begin{bmatrix} 1 & 1/2 \\ 1/2 & 1 \end{bmatrix}$

(d) In the middle case above, to what *direction* will the $k \to \infty$ behaviour of $\mathbf{x}_k$ tend?

BONUS sketch trajectories for the last case above.

6. [10 points] Consider the matrix $A = \begin{bmatrix} 1 & 1/2 \\ 0 & 1/2 \end{bmatrix}$.

(a) Is this matrix diagonalizable? Why? If so, diagonalize it.

(b) Write an exact expression for the elements of the matrix A^k, where k is any non-negative integer.

7. [10 points]

(a) Orthogonally diagonalize the matrix $A = \begin{bmatrix} 4 & 2 & -3 \\ 2 & 4 & -3 \\ -3 & -3 & 9 \end{bmatrix}$, or explain why it is not possible. That is, try to find an orthogonal matrix P and diagonal matrix D such that $A = PDP^T$.

(b) Find the coordinates of the point $\mathbf{x} = \begin{bmatrix} 1 \\ 2 \\ 0 \end{bmatrix}$ in the eigenvector basis.

8. [12 points]

(a) Consider the singular matrix $M = \begin{bmatrix} a & b & c & d \\ a & b & c & d \\ a & b & c & d \\ a & b & c & d \end{bmatrix}$, where a, b, c, d are any real numbers. Show that det $(A + I) = 1 + a + b + c + d$. [Hint: cofactor won't help you].

(b) Suppose matrices A and B have the *same column space* (they may have different columns though!). Do A and B necessarily have the i) same number of pivots? ii) same nullspace? iii) If A is invertible can you conclude B is too?

9. [8 points] Turn to page 61 of book and solve the network flow problem (Example 2).

Practise questions for new material since Midterm 2 include the following. You could make your own practise exam by picking 4–5 of them, one from each major section, and throwing in some true/false questions from the supplementary exercises, also picking 4–5 from Midterm 1 and 2 material.

I apologize if a couple duplicate HW questions. Also, if one seems too hard for an exam, or you are stuck how to do it, email me to ask.

4.9: 11 (b is asking for long-time limit)
5.3: 3, 5, 17, 25.
5.6: 2, 8.

Chapter 5 supplementary exercises (p. 183): 1, 3.

6.1: 20, 25.
6.2: 11, 13, 17, 27.
6.3: 7, 11, 17.
6.5: 1, 13, 17.

Chapter 6 Supplementary Exercises: 1, 5.

7.1: 7, 9, 13, 19, 25a-c.

Chapter 7 Supplementary Exercises: 1a–c, e, n, 2, 4, 5.

For some (more theoretical but fun) practise questions, see the previous exam linked on the Resources page, at
`http://www.math.dartmouth.edu/~m22f05/math_22_final_practice.pdf`