Math 22 HW 9

6.1 19 (3 points) a. True. See the definition of $\|v\|$.

b. True. See Theorem 1(c).

c. True. See the discussion of Fig. 5.

d. False. Counterexample $\begin{bmatrix} 1 & 1 \\ 0 & 0 \end{bmatrix}$.

e. True. See the box following Example 6.

6.2 10 (2 points) Show $u_1 \cdot u_2 = 0$, $u_2 \cdot u_3 = 0$, $u_3 \cdot u_1 = 0$.

Use Theorem 4 and observe that three linearly independent vectors in $\mathbb{R}^3$ form a basis.

$x = \underset{\alpha_1}{\frac{4}{3}} u_1 + \underset{\alpha_2}{\frac{1}{3}} u_2 + \underset{\alpha_3}{\frac{1}{3}} u_3.$ ← use projection formula $\alpha_i = \frac{\vec{x} \cdot \vec{u}_i}{\vec{u}_i \cdot \vec{u}_i}$ $i = 1,2,3$

14 (2 points) $y = \begin{bmatrix} 14/5 \\ 2/5 \end{bmatrix} + \begin{bmatrix} -4/5 \\ 28/5 \end{bmatrix}$

20 (3 points) Not orthogonal. Orthonormal set $\begin{bmatrix} -2/3 \\ 1/3 \\ 2/3 \end{bmatrix}, \begin{bmatrix} 1/\sqrt{5} \\ 2/\sqrt{5} \\ 0 \end{bmatrix}$

left alone   divided by $\sqrt{5}$

means $\vec{u}_i \cdot \vec{u}_j = \begin{cases} 1 & i = j \\ 0 & i \neq j \end{cases}$ ie $U^T U = I$

28 (2 points) U orthogonal $\Rightarrow$ $UU^{-1} = UU^T = I$.

now use IMT to claim also $UU^T = I$, since $U^{-1} = U^T$.

Theorem 6 $\Rightarrow$ U^T has orthonormal columns.

[In particular, the columns of U^T are linearly independent and hence forms a basis for $\mathbb{R}^n$]

$\therefore$ The rows of U form a ~~basis~~ (an orthonormal basis) for $\mathbb{R}^n$.

note L.I. not relevant here

if $V = U^T$ then $UU^T = I$ means $V^T V = I$ ie V is orthogonal matrix.

6.3 2 (2 points) $v = 2u_1 + \frac{3}{7} u_2 + \frac{12}{7} u_3 - \frac{8}{7} u_4$.

8 (3 points) $y = \begin{bmatrix} -3/2 \\ 7/2 \\ 1 \end{bmatrix} + \begin{bmatrix} -5/2 \\ 1/2 \\ 2 \end{bmatrix}$

24 (3 points) a. By hypothesis, the vectors $w_1, \cdots, w_p$ are pairwise orthogonal, and the vectors $v_1, \cdots, v_q$ are pairwise orthogonal.

Also, $w_i \cdot v_j = 0$ for any i and j because the v's are in the orthogonal complement of W.

b. $\forall y \in \mathbb{R}^n$, write $y = \hat{y} + z$ as in the Orthogonal Decomposition Theorem, with $\hat{y} \in W, z \in W^{\perp}$

Then there exist scalars $c_1, \cdots, c_p$ and $d_1, \cdots, d_q$ such that

$$y = \hat{y} + z = c_1 w_1 + \cdots + c_p w_p + d_1 v_1 + \cdots + d_q v_q \quad \text{for any } \vec{y} \in \mathbb{R}^n.$$

Thus $\{w_1, \cdots, w_p, v_1, \cdots, v_q\}$ spans $\mathbb{R}^n$.

c. The set $\{w_1, \cdots, w_p, v_1, \cdots, v_q\}$ is linearly independent by (a), spans $\mathbb{R}$ by (b), and thus is a basis for $\mathbb{R}^n$. Hence $\dim W + \dim W^{\perp} = p + q = \dim \mathbb{R}^n = n$.

6.5 2 (3 points) a. $\begin{bmatrix} 12 & 8 \\ 8 & 10 \end{bmatrix} \begin{bmatrix} x_1 \\ x_2 \end{bmatrix} = \begin{bmatrix} -24 \\ -2 \end{bmatrix}$ b. $\hat{x} = \begin{bmatrix} -4 \\ 3 \end{bmatrix}$.

20 (3 points) Suppose that $Ax = 0$. Then $A^T A x = A^T 0 = 0$.

Since $A^T A$ is invertible by hypothesis, $x = 0$.

$\therefore$ The columns of A are linearly independent.

7.1 8 (3 points) Orthogonal. $U^{-1} = U^T = \begin{bmatrix} 1/\sqrt{2} & 1/\sqrt{2} \\ -1/\sqrt{2} & 1/\sqrt{2} \end{bmatrix}$ Counterclockwise rotation by $45°$

20 (3 points) $P = \begin{bmatrix} 2/3 & -1/3 & 2/3 \\ -1/3 & 2/3 & 2/3 \\ 2/3 & 2/3 & -1/3 \end{bmatrix}$, $D = \begin{bmatrix} 13 & 0 & 0 \\ 0 & 7 & 0 \\ 0 & 0 & 1 \end{bmatrix}$.

35 (3 points) a. Note that $(uu^T)x = u(u^T x) = (u^T x)u$ because $u^T x$ is a scalar.

$\therefore u^T(x - Bx) = u^T x - u^T(uu^T)x = u^T x - (u^T u)u^T x = u^T x - u^T x = 0$.

$\therefore Bx$ is the orthogonal projection of x onto u.

b. $B_{ij} = u_i u_j = u_j u_i = B_{ji} \Rightarrow B$ is symmetric.

$B^2 = uu^T uu^T = u(u^T u)u^T = uu^T = B$

c. $Bu = uu^T \cdot u = u(u^T u) = u \cdot 1 = 1 \cdot u$

$\therefore u$ is an eigenvector of B with an eigenvalue of 1.