HW8 Math 22

5.3 2 (3points) $\frac{1}{16}\begin{bmatrix} 151 & 90 \\ -225 & -134 \end{bmatrix}$

10 (2points) $P=\begin{bmatrix} 1 & -3 \\ 1 & 4 \end{bmatrix}$, $D=\begin{bmatrix} 5 & 0 \\ 0 & -2 \end{bmatrix}$

12 (3points) $P=\begin{bmatrix} 1 & -1 & -1 \\ 1 & 1 & 0 \\ 1 & 0 & 1 \end{bmatrix}$, $D=\begin{bmatrix} 8 & 0 & 0 \\ 0 & 2 & 0 \\ 0 & 0 & 2 \end{bmatrix}$.

26 (2points) Yes, if the third eigenspace is only one-dimensional. In this case, the sum of the dimensions of the eigenspaces will be six, whereas the matrix is 7×7. See Theorem 7(b).

28 (2points). If A has n linearly independent eigenvectors, then by the Diagonalization Theorem, $A=PDP^{-1}$ for some invertible P and diagonal D.

Then $A^T=(PDP^{-1})^T=(P^{-1})^TD^TP^T=(P^T)^{-1}DP^T=QDQ^{-1}$, where $Q=(P^T)^{-1}$.

Thus A is diagonalizable.

By the Diagonalization Theorem, the columns of Q are n linearly independent eigenvectors of A^T.

4.9 12 (3points) Each food will be preferred equally, because $\begin{bmatrix} 1/3 \\ 1/3 \\ 1/3 \end{bmatrix}$ is the steady-state vector.

17 (3points) a. The entries in a column of P sum to 1. A column in the matrix $P-I$ has the same entries as in P except that one of the entries is decreased by 1.

Hence each column sum is 0.

b. By (a), the bottom row of $P-I$ is the negative of the sum of other rows.

c. By (b) and the Spanning Set Theorem, the bottom row of $P-I$ can be removed and the remaining $(n-1)$ rows will still span the row space.

d. By the Rank Theorem and (c), the dimension of the column space of $P-I$ is less than n. Hence the null space is nontrivial.

Google:

using null(P − eye(3), 'r') then dividing by the sum of the column vec.

A (1 points). $P=\begin{bmatrix} 0 & 1/2 & 1 \\ 1 & 0 & 0 \\ 0 & 1/2 & 0 \end{bmatrix}$. Steady state $\begin{bmatrix} 0.4 \\ 0.4 \\ 0.2 \end{bmatrix}$

∴ a and b have the highest rank. (jointly highest).

2 (1 point). It will increase, approching 1. ← so b can beat the system this way.

3 (2 point). $P=\begin{bmatrix} 0 & 1/2 & 0 \\ 1 & 0 & 0 \\ 0 & 1/2 & 1 \end{bmatrix}$. Steady state $\begin{bmatrix} 0 \\ 0 \\ 1 \end{bmatrix}$, ie c gets all the ranking, others are zero!

This is not fair!

4 (2 point) Steady state $\begin{bmatrix} 0.1115 \\ 0.1448 \\ 0.7436 \end{bmatrix}$

$S := \text{sum}(\vec{x})$, unknown.

5. (BONUS): if $\vec{x} = dP\vec{x} + (1-d)\vec{e}$ the column sums are $S = dS + (1-d)1$. (~~since $P\vec{x}$ has same sum as $\vec{x}$~~.)

$\Rightarrow S(1-d) = 1-d \Rightarrow S = 1$ ✓.

5.6 12 (3 points) Saddle point. Eigenvalues: 1.1, 0.8. Greatest repulsion: line through (0,0) and (1,1).

Greatest attraction: line through (0,0) and (2,1).

6.1 12 (2 points) $\begin{bmatrix} 0.8 \\ 0.6 \end{bmatrix}$

14 (3 points) $2\sqrt{17}$.

Note on eigenvector issue — is it a lin. system or an eigenvector??

$$\underbrace{(I - dP)}_{\text{"}A\text{"}}\vec{x} = \underbrace{(1-d)\vec{e}}_{\text{"}\vec{b}\text{"}} \quad \text{is lin. system } A\vec{x}=\vec{b}.$$

But $\vec{e}$ can be written $M\vec{x}$ where $M = \begin{bmatrix} 1/n & 1/n & \cdots \\ 1/n & 1/n & \cdots \\ \vdots & & \end{bmatrix}$, since $\vec{x}$ is a prob. vector

$$\Rightarrow (I - dP)\vec{x} = (1-d)M\vec{x}$$

$$\Rightarrow A\vec{x} = \vec{x} \quad \text{with modified stochastic matrix } A = \underbrace{dP + (1-d)M}_{\text{a 'convex' combination of } P, M}$$

Thus it is also an eigenvector problem!

Note the matrix M corresponds to randomly jumping anywhere in the web.