Math 22 Lin Alg: Homework 8

due Wed Aug 16 ...but best if do relevant questions after each lecture

After a bit of diagonalization, we reap the benefits of our work with some cool applications. Note I jump backwards to 4.9 following the lecture order.

5.3: Goals: Be able to diagonalize a matrix or tell it is not diagonalizable. Know how the factorization $A = PDP^{-1}$ gives you A^k efficiently.

2 (note you'll need to get P^{-1}), 8, 10 (key), 12 (key), 26, 28 (fun proof).

4.9: Goals: Be able to construct a stochastic matrix from a description in words, find its steady state vector. Understand the long-term behavior, connection to eigenvalues.

2 (won't need matlab for this),

12 (compute the steady state vector by hand. You can check with matlab),

17 (beautiful proof that $\lambda = 1$ is an eigenvalue of P. Make sure you explain precisely each step).

A. Fun exploration of the Google PageRank algorithm (also see links on our Resources webpage). Imagine the whole web has $n = 3$ pages: a, b, c. Page a links to b; page b links to a and c; page c links to a.

1. Draw the graph, write down the stochastic matrix P, and find its steady-state vector with matlab [Hint: `[V,D] = eig(P)` gives `V` the matrix with eigenvectors in columns]. Which page(s) have highest rank?
2. Page b's webmaster tries to beat the system by adding a link from b to itself. Does b's rank increase? What happens if b adds more and more links to itself? (try it)
3. Going back to the original P, now c decides to link once to itself, but not to a. What happens to the steady-state vector? Does this seem fair? Explain it in terms of the 'random surfer' model.
4. The problem just exposed is fixed in PageRank as follows: the $n \times n$ eigenvector equation $\mathbf{x} = P\mathbf{x}$ is modified to

$$\mathbf{x} = dP\mathbf{x} + (1-d)\mathbf{e}, \qquad \text{with } 0 < d < 1, \quad \text{and the fixed column vector } \mathbf{e} = [1/n \; 1/n \; \ldots \; 1/n]^T$$

With $d = 0.85$, a typical choice [1], find the steady state solution $\mathbf{x}$ for the previous situation. [Hint: rearrange it algebraically—is it, as claimed in the original paper (!), still an eigenvector equation? Then use matlab to solve it numerically]. *Congratulations: you now understand (part of) Google!* In practise $n \sim 10^{10}$. Try to imagine that. Do you think an $O(n^3)$ algorithm is used?

5. [Optional bonus]: Prove that any $\mathbf{x}$ solving the modified equation is still a probability vector.

5.6: Goals: Be able to evolve a dynamical system via the eigenvector decomposition, predict long-time behavior by finding attractors, repellors, or saddle points.

1, 12 (feel free to use matlab to check your eigenvalues and vectors).

6.1: Goals: Compute the inner product of vectors and how it relates to the length; compute the distance between two vectors; how to determine when vectors are orthogonal, and how it relates to the Pythagorean theorem.

2, 8, 12, 14, 16. (These are all very quick—do not be alarmed).

[1] See web references. Also note: to connect with stochastic matrices, I have chosen our definition of page rank to be n times smaller than *PR* discussed in Google literature.