1.3 EXERCISES

In Exercises 1 and 2, compute $\mathbf{u} + \mathbf{v}$ and $\mathbf{u} - 2\mathbf{v}$.

1. $\mathbf{u} = \begin{bmatrix} -1 \\ 2 \end{bmatrix}, \mathbf{v} = \begin{bmatrix} -3 \\ -1 \end{bmatrix}$ 2. $\mathbf{u} = \begin{bmatrix} 3 \\ 2 \end{bmatrix}, \mathbf{v} = \begin{bmatrix} 2 \\ -1 \end{bmatrix}$

In Exercises 3 and 4, display the following vectors using arrows on an xy-graph: $\mathbf{u}, \mathbf{v}, -\mathbf{v}, -2\mathbf{v}, \mathbf{u} + \mathbf{v}, \mathbf{u} - \mathbf{v},$ and $\mathbf{u} - 2\mathbf{v}$. Notice that $\mathbf{u} - \mathbf{v}$ is the vertex of a parallelogram whose other vertices are $\mathbf{u}, \mathbf{0}$, and $-\mathbf{v}$.

3. $\mathbf{u}$ and $\mathbf{v}$ as in Exercise 1   4. $\mathbf{u}$ and $\mathbf{v}$ as in Exercise 2

In Exercises 5 and 6, write a system of equations that is equivalent to the given vector equation.

5. $x_1 \begin{bmatrix} 6 \\ -1 \\ 5 \end{bmatrix} + x_2 \begin{bmatrix} -3 \\ 4 \\ 0 \end{bmatrix} = \begin{bmatrix} 1 \\ -7 \\ -5 \end{bmatrix}$

6. $x_1 \begin{bmatrix} -2 \\ 3 \end{bmatrix} + x_2 \begin{bmatrix} 8 \\ 5 \end{bmatrix} + x_3 \begin{bmatrix} 1 \\ -6 \end{bmatrix} = \begin{bmatrix} 0 \\ 0 \end{bmatrix}$

Use the accompanying figure to write each vector listed in Exercises 7 and 8 as a linear combination of $\mathbf{u}$ and $\mathbf{v}$. Is every vector in $\mathbb{R}^2$ a linear combination of $\mathbf{u}$ and $\mathbf{v}$?
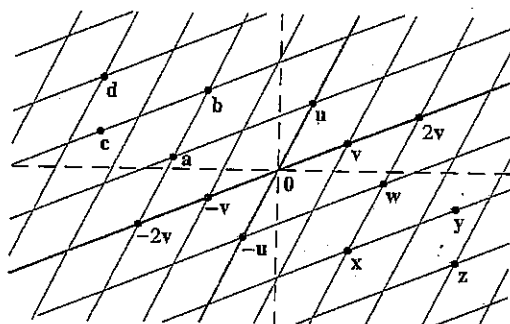
7. Vectors $\mathbf{a}, \mathbf{b}, \mathbf{c}$, and $\mathbf{d}$ 8. Vectors $\mathbf{w}, \mathbf{x}, \mathbf{y}$, and $\mathbf{z}$

In Exercises 9 and 10, write a vector equation that is equivalent to the given system of equations.

9.
$$\begin{aligned} x_2 + 5x_3 &= 0 \\ 4x_1 + 6x_2 - x_3 &= 0 \\ -x_1 + 3x_2 - 8x_3 &= 0 \end{aligned}$$

10.
$$\begin{aligned} 4x_1 + x_2 + 3x_3 &= 9 \\ x_1 - 7x_2 - 2x_3 &= 2 \\ 8x_1 + 6x_2 - 5x_3 &= 15 \end{aligned}$$

In Exercises 11 and 12, determine if $\mathbf{b}$ is a linear combination of $\mathbf{a}_1, \mathbf{a}_2$, and $\mathbf{a}_3$.

11. $a_1 = \begin{bmatrix} 1 \\ -2 \\ 0 \end{bmatrix}$, $a_2 = \begin{bmatrix} 0 \\ 1 \\ 2 \end{bmatrix}$, $a_3 = \begin{bmatrix} 5 \\ -6 \\ 8 \end{bmatrix}$, $b = \begin{bmatrix} 2 \\ -1 \\ 6 \end{bmatrix}$

12. $a_1 = \begin{bmatrix} 1 \\ -2 \\ 2 \end{bmatrix}$, $a_2 = \begin{bmatrix} 0 \\ 5 \\ 5 \end{bmatrix}$, $a_3 = \begin{bmatrix} 2 \\ 0 \\ 8 \end{bmatrix}$, $b = \begin{bmatrix} -5 \\ 11 \\ -7 \end{bmatrix}$

In Exercises 13 and 14, determine if b is a linear combination of the vectors formed from the columns of the matrix A.

13. $A = \begin{bmatrix} 1 & -4 & 2 \\ 0 & 3 & 5 \\ -2 & 8 & -4 \end{bmatrix}$, $b = \begin{bmatrix} 3 \\ -7 \\ -3 \end{bmatrix}$

14. $A = \begin{bmatrix} 1 & -2 & -6 \\ 0 & 3 & 7 \\ 1 & -2 & 5 \end{bmatrix}$, $b = \begin{bmatrix} 11 \\ -5 \\ 9 \end{bmatrix}$

In Exercises 15 and 16, list five vectors in Span $\{v_1, v_2\}$. For each vector, show the weights on v_1 and v_2 used to generate the vector and list the three entries of the vector. Do not make a sketch.

15. $v_1 = \begin{bmatrix} 7 \\ 1 \\ -6 \end{bmatrix}$, $v_2 = \begin{bmatrix} -5 \\ 3 \\ 0 \end{bmatrix}$

16. $v_1 = \begin{bmatrix} 3 \\ 0 \\ 2 \end{bmatrix}$, $v_2 = \begin{bmatrix} -2 \\ 0 \\ 3 \end{bmatrix}$

17. Let $a_1 = \begin{bmatrix} 1 \\ 4 \\ -2 \end{bmatrix}$, $a_2 = \begin{bmatrix} -2 \\ -3 \\ 7 \end{bmatrix}$, and $b = \begin{bmatrix} 4 \\ 1 \\ h \end{bmatrix}$. For what value(s) of h is b in the plane spanned by a_1 and a_2?

18. Let $v_1 = \begin{bmatrix} 1 \\ 0 \\ -2 \end{bmatrix}$, $v_2 = \begin{bmatrix} -3 \\ 1 \\ 8 \end{bmatrix}$, and $y = \begin{bmatrix} h \\ -5 \\ -3 \end{bmatrix}$. For what value(s) of h is y in the plane generated by v_1 and v_2?

19. Give a geometric description of Span $\{v_1, v_2\}$ for the vectors
$v_1 = \begin{bmatrix} 8 \\ 2 \\ -6 \end{bmatrix}$ and $v_2 = \begin{bmatrix} 12 \\ 3 \\ -9 \end{bmatrix}$.

20. Give a geometric description of Span $\{v_1, v_2\}$ for the vectors in Exercise 16.

21. Let $u = \begin{bmatrix} 2 \\ -1 \end{bmatrix}$ and $v = \begin{bmatrix} 2 \\ 1 \end{bmatrix}$. Show that $\begin{bmatrix} h \\ k \end{bmatrix}$ is in Span $\{u, v\}$ for all h and k.

22. Construct a 3×3 matrix A, with nonzero entries, and a vector b in $\mathbb{R}^3$ such that b is *not* in the set spanned by the columns of A.

In Exercises 23 and 24, mark each statement True or False. Justify each answer.

23. a. Another notation for the vector $\begin{bmatrix} -4 \\ 3 \end{bmatrix}$ is $[\,-4 \quad 3\,]$.

b. The points in the plane corresponding to $\begin{bmatrix} -2 \\ 5 \end{bmatrix}$ and $\begin{bmatrix} -5 \\ 2 \end{bmatrix}$ lie on a line through the origin.

c. An example of a linear combination of vectors v_1 and v_2 is the vector $\frac{1}{2}v_1$.

d. The solution set of the linear system whose augmented matrix is $[\,a_1 \quad a_2 \quad a_3 \quad b\,]$ is the same as the solution set of the equation $x_1 a_1 + x_2 a_2 + x_3 a_3 = b$.

e. The set Span $\{u, v\}$ is always visualized as a plane through the origin.

24. a. Any list of five real numbers is a vector in $\mathbb{R}^5$.

b. The vector u results when a vector $u - v$ is added to the vector v.

c. The weights $c_1, \dots, c_p$ in a linear combination $c_1 v_1 + \cdots + c_p v_p$ cannot all be zero.

d. When u and v are nonzero vectors, Span $\{u, v\}$ contains the line through u and the origin.

e. Asking whether the linear system corresponding to an augmented matrix $[\,a_1 \quad a_2 \quad a_3 \quad b\,]$ has a solution amounts to asking whether b is in Span $\{a_1, a_2, a_3\}$.

25. Let $A = \begin{bmatrix} 1 & 0 & -4 \\ 0 & 3 & -2 \\ -2 & 6 & 3 \end{bmatrix}$ and $b = \begin{bmatrix} 4 \\ 1 \\ -4 \end{bmatrix}$. Denote the columns of A by a_1, a_2, a_3, and let $W = $ Span $\{a_1, a_2, a_3\}$.

a. Is b in $\{a_1, a_2, a_3\}$? How many vectors are in $\{a_1, a_2, a_3\}$?

b. Is b in W? How many vectors are in W?

c. Show that a_1 is in W. [*Hint:* Row operations are unnecessary.]

26. Let $A = \begin{bmatrix} 2 & 0 & 6 \\ -1 & 8 & 5 \\ 1 & -2 & 1 \end{bmatrix}$, let $b = \begin{bmatrix} 10 \\ 3 \\ 3 \end{bmatrix}$, and let W be the set of all linear combinations of the columns of A.

a. Is b in W?

b. Show that the third column of A is in W.

27. A mining company has two mines. One day's operation at mine #1 produces ore that contains 20 metric tons of copper

and 550 kilograms of silver, while one day's operation at mine #2 produces ore that contains 30 metric tons of copper and 500 kilograms of silver. Let $v_1 = \begin{bmatrix} 20 \\ 550 \end{bmatrix}$ and $v_2 = \begin{bmatrix} 30 \\ 500 \end{bmatrix}$. Then v_1 and v_2 represent the "output per day" of mine #1 and mine #2, respectively.

a. What physical interpretation can be given to the vector $5v_1$?

b. Suppose the company operates mine #1 for x_1 days and mine #2 for x_2 days. Write a vector equation whose solution gives the number of days each mine should operate in order to produce 150 tons of copper and 2825 kilograms of silver. Do not solve the equation.

c. [M] Solve the equation in (b).

28. A steam plant burns two types of coal: anthracite (A) and bituminous (B). For each ton of A burned, the plant produces 27.6 million Btu of heat, 3100 grams (g) of sulfur dioxide, and 250 g of particulate matter (solid-particle pollutants). For each ton of B burned, the plant produces 30.2 million Btu, 6400 g of sulfur dioxide, and 360 g of particulate matter.

a. How much heat does the steam plant produce when it burns x_1 tons of A and x_2 tons of B?

b. Suppose the output of the steam plant is described by a vector that lists the amounts of heat, sulfur dioxide, and particulate matter. Express this output as a linear combination of two vectors, assuming that the plant burns x_1 tons of A and x_2 tons of B.

c. [M] Over a certain time period, the steam plant produced 162 million Btu of heat, 23,610 g of sulfur dioxide, and 1623 g of particulate matter. Determine how many tons of each type of coal the steam plant must have burned. Include a vector equation as part of your solution.

29. Let $v_1, \ldots, v_k$ be points in $\mathbb{R}^3$ and suppose that for $j = 1, \ldots, k$ an object with mass m_j is located at point v_j. Physicists call such objects *point masses*. The total mass of the system of point masses is

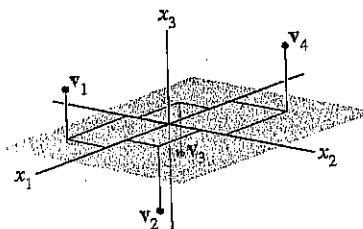$$m = m_1 + \cdots + m_k$$

The *center of gravity* (or *center of mass*) of the system is

$$\bar{v} = \frac{1}{m}[m_1 v_1 + \cdots + m_k v_k]$$

Compute the center of gravity of the system consisting of the following point masses (see the figure):

Point	Mass
$v_1 = (5, -4, 3)$	2 g
$v_2 = (4, 3, -2)$	5 g
$v_3 = (-4, -3, -1)$	2 g
$v_4 = (-9, 8, 6)$	1 g

30. Let v be the center of mass of a system of point masses located at $v_1, \ldots, v_k$ as in Exercise 29. Is v in Span $\{v_1, \ldots, v_k\}$? Explain.

31. A thin triangular plate of uniform density and thickness has vertices at $v_1 = (0, 1)$, $v_2 = (8, 1)$, and $v_3 = (2, 4)$, as in the figure below, and the mass of the plate is 3 g.
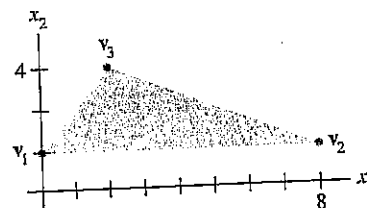
a. Find the (x, y)-coordinates of the center of mass of the plate. This "balance point" of the plate coincides with the center of mass of a system consisting of three 1-gram point masses located at the vertices of the plate.

b. Determine how to distribute an additional mass of 6 g at the three vertices of the plate to move the balance point of the plate to (2, 2). [*Hint:* Let w_1, w_2, and w_3 denote the masses added at the three vertices, so that $w_1 + w_2 + w_3 = 6$.]

32. Consider the vectors v_1, v_2, v_3 and b in $\mathbb{R}^2$, shown in the figure. Does the equation $x_1v_1 + x_2v_2 + x_3v_3 = b$ have a solution? Is the solution unique? Use the figure to explain your answers.
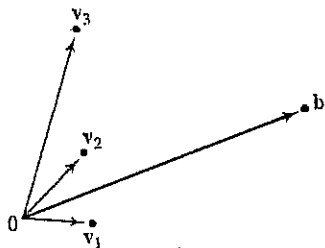
33. Use the vectors $u = (u_1, \ldots, u_n)$, $v = (v_1, \ldots, v_n)$, and $w = (w_1, \ldots, w_n)$ to verify the following algebraic properties of $\mathbb{R}^n$.

a. $(u + v) + w = u + (v + w)$

b. $c(u + v) = cu + cv$ for each scalar c

34. Use the vector $u = (u_1, \ldots, u_n)$ to verify the following algebraic properties of $\mathbb{R}^n$.

a. $u + (-u) = (-u) + u = 0$

b. $c(du) = (cd)u$ for all scalars c and d

1.4 EXERCISES

Compute the products in Exercises 1–4 using (a) the definition, as in Example 1, and (b) the row–vector rule for computing $A\mathbf{x}$. If a product is undefined, explain why.

1. $\begin{bmatrix} -4 & 2 \\ 1 & 6 \\ 0 & 1 \end{bmatrix} \begin{bmatrix} 3 \\ -2 \\ 7 \end{bmatrix}$

2. $\begin{bmatrix} 2 \\ 6 \\ -1 \end{bmatrix} \begin{bmatrix} 5 \\ -1 \end{bmatrix}$

3. $\begin{bmatrix} 6 & 5 \\ -4 & -3 \\ 7 & 6 \end{bmatrix} \begin{bmatrix} 2 \\ -3 \end{bmatrix}$

4. $\begin{bmatrix} 8 & 3 & -4 \\ 5 & 1 & 2 \end{bmatrix} \begin{bmatrix} 1 \\ 1 \\ 1 \end{bmatrix}$

In Exercises 5–8, use the definition of $A\mathbf{x}$ to write the matrix equation as a vector equation, or vice versa.

5. $\begin{bmatrix} 5 & 1 & -8 & 4 \\ -2 & -7 & 3 & -5 \end{bmatrix} \begin{bmatrix} 5 \\ -1 \\ 3 \\ -2 \end{bmatrix} = \begin{bmatrix} -8 \\ 16 \end{bmatrix}$

6. $\begin{bmatrix} 7 & -3 \\ 2 & 1 \\ 9 & -6 \\ -3 & 2 \end{bmatrix} \begin{bmatrix} -2 \\ -5 \end{bmatrix} = \begin{bmatrix} 1 \\ -9 \\ 12 \\ -4 \end{bmatrix}$

7. $x_1 \begin{bmatrix} 4 \\ -1 \\ 7 \\ -4 \end{bmatrix} + x_2 \begin{bmatrix} -5 \\ 3 \\ -5 \\ 1 \end{bmatrix} + x_3 \begin{bmatrix} 7 \\ -8 \\ 0 \\ 2 \end{bmatrix} = \begin{bmatrix} 6 \\ -8 \\ 0 \\ -7 \end{bmatrix}$

8. $z_1 \begin{bmatrix} 4 \\ -2 \end{bmatrix} + z_2 \begin{bmatrix} -4 \\ 5 \end{bmatrix} + z_3 \begin{bmatrix} -5 \\ 4 \end{bmatrix} + z_4 \begin{bmatrix} 3 \\ 0 \end{bmatrix} = \begin{bmatrix} 4 \\ 13 \end{bmatrix}$

In Exercises 9 and 10, write the system first as a vector equation and then as a matrix equation.

9. $\begin{aligned} 3x_1 + x_2 - 5x_3 &= 9 \\ x_2 + 4x_3 &= 0 \end{aligned}$

10. $\begin{aligned} 8x_1 - x_2 &= 4 \\ 5x_1 + 4x_2 &= 1 \\ x_1 - 3x_2 &= 2 \end{aligned}$

Given A and $\mathbf{b}$ in Exercises 11 and 12, write the augmented matrix for the linear system that corresponds to the matrix equation $A\mathbf{x} = \mathbf{b}$. Then solve the system and write the solution as a vector.

11. $A = \begin{bmatrix} 1 & 2 & 4 \\ 0 & 1 & 5 \\ -2 & -4 & -3 \end{bmatrix}, \mathbf{b} = \begin{bmatrix} -2 \\ 2 \\ 9 \end{bmatrix}$

12. $A = \begin{bmatrix} 1 & 2 & 1 \\ -3 & -1 & 2 \\ 0 & 5 & 3 \end{bmatrix}, \mathbf{b} = \begin{bmatrix} 0 \\ 1 \\ -1 \end{bmatrix}$

13. Let $\mathbf{u} = \begin{bmatrix} 0 \\ 4 \\ 4 \end{bmatrix}$ and $A = \begin{bmatrix} 3 & -5 \\ -2 & 6 \\ 1 & 1 \end{bmatrix}$. Is $\mathbf{u}$ in the plane $\mathbb{R}^3$ spanned by the columns of A? (See the figure.) Why or why not?
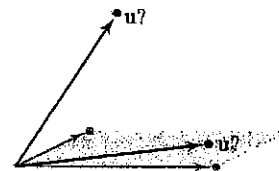
Where is $\mathbf{u}$?

14. Let $u = \begin{bmatrix} 2 \\ -3 \\ 2 \end{bmatrix}$ and $A = \begin{bmatrix} 5 & 8 & 7 \\ 0 & 1 & -1 \\ 1 & 3 & 0 \end{bmatrix}$. Is u in the subset of $\mathbb{R}^3$ spanned by the columns of A? Why or why not?

15. Let $A = \begin{bmatrix} 2 & -1 \\ -6 & 3 \end{bmatrix}$ and $b = \begin{bmatrix} b_1 \\ b_2 \end{bmatrix}$. Show that the equation $Ax = b$ does not have a solution for all possible b, and describe the set of all b for which $Ax = b$ *does* have a solution.

16. Repeat Exercise 15: $A = \begin{bmatrix} 1 & -3 & -4 \\ -3 & 2 & 6 \\ 5 & -1 & -8 \end{bmatrix}$, $b = \begin{bmatrix} b_1 \\ b_2 \\ b_3 \end{bmatrix}$.

Exercises 17–20 refer to the matrices A and B below. Make appropriate calculations that justify your answers and mention an appropriate theorem.

$$A = \begin{bmatrix} 1 & 3 & 0 & 3 \\ -1 & -1 & -1 & 1 \\ 0 & -4 & 2 & -8 \\ 2 & 0 & 3 & -1 \end{bmatrix} \quad B = \begin{bmatrix} 1 & 3 & -2 & 2 \\ 0 & 1 & 1 & -5 \\ 1 & 2 & -3 & 7 \\ -2 & -8 & 2 & -1 \end{bmatrix}$$

17. How many rows of A contain a pivot position? Does the equation $Ax = b$ have a solution for each b in $\mathbb{R}^4$?

18. Do the columns of B span $\mathbb{R}^4$? Does the equation $Bx = y$ have a solution for each y in $\mathbb{R}^4$?

19. Can each vector in $\mathbb{R}^4$ be written as a linear combination of the columns of the matrix A above? Do the columns of A span $\mathbb{R}^4$?

20. Can every vector in $\mathbb{R}^4$ be written as a linear combination of the columns of the matrix B above? Do the columns of B span $\mathbb{R}^3$?

21. Let $v_1 = \begin{bmatrix} 1 \\ 0 \\ -1 \\ 0 \end{bmatrix}$, $v_2 = \begin{bmatrix} 0 \\ -1 \\ 0 \\ 1 \end{bmatrix}$, $v_3 = \begin{bmatrix} 1 \\ 0 \\ 0 \\ -1 \end{bmatrix}$.

Does $\{v_1, v_2, v_3\}$ span $\mathbb{R}^4$? Why or why not?

22. Let $v_1 = \begin{bmatrix} 0 \\ 0 \\ -2 \end{bmatrix}$, $v_2 = \begin{bmatrix} 0 \\ -3 \\ 8 \end{bmatrix}$, $v_3 = \begin{bmatrix} 4 \\ -1 \\ -5 \end{bmatrix}$.

Does $\{v_1, v_2, v_3\}$ span $\mathbb{R}^3$? Why or why not?

In Exercises 23 and 24, mark each statement True or False. Justify each answer.

23. a. The equation $Ax = b$ is referred to as a *vector equation*.

   b. A vector b is a linear combination of the columns of a matrix A if and only if the equation $Ax = b$ has at least one solution.

c. The equation $Ax = b$ is consistent if the augmented matrix $[A \quad b]$ has a pivot position in every row.

d. The first entry in the product Ax is a sum of products.

e. If the columns of an $m \times n$ matrix A span $\mathbb{R}^m$, then the equation $Ax = b$ is consistent for each b in $\mathbb{R}^m$.

f. If A is an $m \times n$ matrix and if the equation $Ax = b$ is inconsistent for some b in $\mathbb{R}^m$, then A cannot have a pivot position in every row.

24. a. Every matrix equation $Ax = b$ corresponds to a vector equation with the same solution set.

   b. Any linear combination of vectors can always be written in the form Ax for a suitable matrix A and vector x.

   c. The solution set of a linear system whose augmented matrix is $[a_1 \quad a_2 \quad a_3 \quad b]$ is the same as the solution set of $Ax = b$, if $A = [a_1 \quad a_2 \quad a_3]$.

   d. If the equation $Ax = b$ is inconsistent, then b is not in the set spanned by the columns of A.

   e. If the augmented matrix $[A \quad b]$ has a pivot position in every row, then the equation $Ax = b$ is inconsistent.

   f. If A is an $m \times n$ matrix whose columns do not span $\mathbb{R}^m$, then the equation $Ax = b$ is inconsistent for some b in $\mathbb{R}^m$.

25. Note that $\begin{bmatrix} 4 & -3 & 1 \\ 5 & -2 & 5 \\ -6 & 2 & -3 \end{bmatrix} \begin{bmatrix} -3 \\ -1 \\ 2 \end{bmatrix} = \begin{bmatrix} -7 \\ -3 \\ 10 \end{bmatrix}$. Use this fact (and no row operations) to find scalars c_1, c_2, c_3 such that $\begin{bmatrix} -7 \\ -3 \\ 10 \end{bmatrix} = c_1 \begin{bmatrix} 4 \\ 5 \\ -6 \end{bmatrix} + c_2 \begin{bmatrix} -3 \\ -2 \\ 2 \end{bmatrix} + c_3 \begin{bmatrix} 1 \\ 5 \\ -3 \end{bmatrix}$.

26. Let $u = \begin{bmatrix} 7 \\ 2 \\ 5 \end{bmatrix}$, $v = \begin{bmatrix} 3 \\ 1 \\ 3 \end{bmatrix}$, and $w = \begin{bmatrix} 6 \\ 1 \\ 0 \end{bmatrix}$.

It can be shown that $3u - 5v - w = 0$. Use this fact (and no row operations) to find x_1 and x_2 that satisfy the equation $\begin{bmatrix} 7 & 3 \\ 2 & 1 \\ 5 & 3 \end{bmatrix} \begin{bmatrix} x_1 \\ x_2 \end{bmatrix} = \begin{bmatrix} 6 \\ 1 \\ 0 \end{bmatrix}$.

27. Let q_1, q_2, q_3, and v represent vectors in $\mathbb{R}^5$, and let x_1, x_2, and x_3 denote scalars. Write the following vector equation as a matrix equation. Identify any symbols you choose to use.

$x_1 q_1 + x_2 q_2 + x_3 q_3 = v$

28. Rewrite the (numerical) matrix equation below in symbolic form as a vector equation, using symbols v_1, v_2, ... for the

vectors and $c_1, c_2, \ldots$ for scalars. Define what each symbol represents, using the data given in the matrix equation.

$$\begin{bmatrix} -3 & 5 & -4 & 9 & 7 \\ 5 & 8 & 1 & -2 & -4 \end{bmatrix} \begin{bmatrix} -3 \\ 2 \\ 4 \\ -1 \\ 2 \end{bmatrix} = \begin{bmatrix} 8 \\ -1 \end{bmatrix}$$

29. Construct a 3×3 matrix, not in echelon form, whose columns span $\mathbb{R}^3$. Show that the matrix you construct has the desired property.

30. Construct a 3×3 matrix, not in echelon form, whose columns do *not* span $\mathbb{R}^3$. Show that the matrix you construct has the desired property.

31. Let A be a 3×2 matrix. Explain why the equation $Ax = b$ cannot be consistent for all b in $\mathbb{R}^3$. Generalize your argument to the case of an arbitrary A with more rows than columns.

32. Could a set of three vectors in $\mathbb{R}^4$ span all of $\mathbb{R}^4$? Explain. What about n vectors in $\mathbb{R}^m$ when n is less than m?

33. Suppose A is a 4×3 matrix and b is a vector in $\mathbb{R}^4$ with the property that $Ax = b$ has a unique solution. What can you say about the reduced echelon form of A? Justify your answer.

34. Suppose A is a 3×3 matrix and b is a vector in $\mathbb{R}^3$ with the property that $Ax = b$ has a unique solution. Explain why the columns of A must span $\mathbb{R}^3$.

35. Let A be a 3×4 matrix, let y_1 and y_2 be vectors in $\mathbb{R}^3$, and let $w = y_1 + y_2$. Suppose $y_1 = Ax_1$ and $y_2 = Ax_2$ for some vec-

tors x_1 and x_2 in $\mathbb{R}^4$. What fact allows you to conclude that the system $Ax = w$ is consistent? (*Note*: x_1 and x_2 denote vectors, not scalar entries in vectors.)

36. Let A be a 5×3 matrix, let y be a vector in $\mathbb{R}^3$, and let z be a vector in $\mathbb{R}^5$. Suppose $Ay = z$. What fact allows you to conclude that the system $Ax = 4z$ is consistent?

[M] In Exercises 37–40, determine if the columns of the matrix span $\mathbb{R}^4$.

37. $\begin{bmatrix} 7 & 2 & -5 & 8 \\ -5 & -3 & 4 & -9 \\ 6 & 10 & -2 & 7 \\ -7 & 9 & 2 & 15 \end{bmatrix}$ **38.** $\begin{bmatrix} 5 & -7 & -4 & 9 \\ 6 & -8 & -7 & 5 \\ 4 & -4 & -9 & -9 \\ -9 & 11 & 16 & 7 \end{bmatrix}$

39. $\begin{bmatrix} 12 & -7 & 11 & -9 & 5 \\ -9 & 4 & -8 & 7 & -3 \\ -6 & 11 & -7 & 3 & -9 \\ 4 & -6 & 10 & -5 & 12 \end{bmatrix}$

40. $\begin{bmatrix} 8 & 11 & -6 & -7 & 13 \\ -7 & -8 & 5 & 6 & -9 \\ 11 & 7 & -7 & -9 & -6 \\ -3 & 4 & 1 & 8 & 7 \end{bmatrix}$

41. [M] Find a column of the matrix in Exercise 39 that can be deleted and yet have the remaining matrix columns still span $\mathbb{R}^4$.

42. [M] Find a column of the matrix in Exercise 40 that can be deleted and yet have the remaining matrix columns still span $\mathbb{R}^4$. Can you delete more than one column?

1.5   EXERCISES

In Exercises 1–4, determine if the system has a nontrivial solution. Try to use as few row operations as possible.

1.
$$2x_1 - 5x_2 + 8x_3 = 0$$
$$-2x_1 - 7x_2 + x_3 = 0$$
$$4x_1 + 2x_2 + 7x_3 = 0$$

2.
$$x_1 - 3x_2 + 7x_3 = 0$$
$$-2x_1 + x_2 - 4x_3 = 0$$
$$x_1 + 2x_2 + 9x_3 = 0$$

3.
$$-3x_1 + 5x_2 - 7x_3 = 0$$
$$-6x_1 + 7x_2 + x_3 = 0$$

4.
$$-5x_1 + 7x_2 + 9x_3 = 0$$
$$x_1 - 2x_2 + 6x_3 = 0$$

In Exercises 5 and 6, follow the method of Examples 1 and 2 to write the solution set of the given homogeneous system in parametric vector form.

5.
$$x_1 + 3x_2 + x_3 = 0$$
$$-4x_1 - 9x_2 + 2x_3 = 0$$
$$- 3x_2 - 6x_3 = 0$$

6.
$$x_1 + 3x_2 - 5x_3 = 0$$
$$x_1 + 4x_2 - 8x_3 = 0$$
$$-3x_1 - 7x_2 + 9x_3 = 0$$

In Exercises 7–12, describe all solutions of $Ax = 0$ in parametric vector form, where A is row equivalent to the given matrix.

7. $\begin{bmatrix} 1 & 3 & -3 & 7 \\ 0 & 1 & -4 & 5 \end{bmatrix}$

8. $\begin{bmatrix} 1 & -2 & -9 & 5 \\ 0 & 1 & 2 & -6 \end{bmatrix}$

9. $\begin{bmatrix} 3 & -9 & 6 \\ -1 & 3 & -2 \end{bmatrix}$

10. $\begin{bmatrix} 1 & 3 & 0 & -4 \\ 2 & 6 & 0 & -8 \end{bmatrix}$

11. $\begin{bmatrix} 1 & -4 & -2 & 0 & 3 & -5 \\ 0 & 0 & 1 & 0 & 0 & -1 \\ 0 & 0 & 0 & 0 & 1 & -4 \\ 0 & 0 & 0 & 0 & 0 & 0 \end{bmatrix}$

12. $\begin{bmatrix} 1 & 5 & 2 & -6 & 9 & 0 \\ 0 & 0 & 1 & -7 & 4 & -8 \\ 0 & 0 & 0 & 0 & 0 & 1 \\ 0 & 0 & 0 & 0 & 0 & 0 \end{bmatrix}$

13. Suppose the solution set of a certain system of linear equations can be described as $x_1 = 5 + 4x_3$, $x_2 = -2 - 7x_3$, with x_3 free. Use vectors to describe this set as a line in $\mathbb{R}^3$.

14. Suppose the solution set of a certain system of linear equations can be described as $x_1 = 3x_4$, $x_2 = 8 + x_4$, $x_3 = 2 - 5x_4$, with x_4 free. Use vectors to describe this set as a "line" in $\mathbb{R}^4$.

15. Follow the method of Example 3 to describe the solutions of the following system in parametric vector form. Also, give a geometric description of the solution set and compare it to that in Exercise 5.

$$x_1 + 3x_2 + x_3 = 1$$
$$-4x_1 - 9x_2 + 2x_3 = -1$$
$$- 3x_2 - 6x_3 = -3$$

16. As in Exercise 15, describe the solutions of the following system in parametric vector form, and provide a geometric comparison with the solution set in Exercise 6.

$$x_1 + 3x_2 - 5x_3 = 4$$
$$x_1 + 4x_2 - 8x_3 = 7$$
$$-3x_1 - 7x_2 + 9x_3 = -6$$

17. Describe and compare the solution sets of $x_1 + 9x_2 - 4x_3 = 0$ and $x_1 + 9x_2 - 4x_3 = -2$.

18. Describe and compare the solution sets of $x_1 - 3x_2 + 5x_3 = 0$ and $x_1 - 3x_2 + 5x_3 = 4$.

In Exercises 19 and 20, find the parametric equation of the line through a parallel to b.

19. $a = \begin{bmatrix} -2 \\ 0 \end{bmatrix}$, $b = \begin{bmatrix} -5 \\ 3 \end{bmatrix}$ **20.** $a = \begin{bmatrix} 3 \\ -4 \end{bmatrix}$, $b = \begin{bmatrix} -7 \\ 8 \end{bmatrix}$

In Exercises 21 and 22, find a parametric equation of the line M through p and q. [*Hint*: M is parallel to the vector $q - p$. See the figure below.]

21. $p = \begin{bmatrix} 2 \\ -5 \end{bmatrix}$, $q = \begin{bmatrix} -3 \\ 1 \end{bmatrix}$ **22.** $p = \begin{bmatrix} -6 \\ 3 \end{bmatrix}$, $q = \begin{bmatrix} 0 \\ -4 \end{bmatrix}$
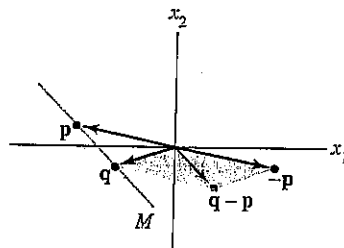
The line through p and q.

In Exercises 23 and 24, mark each statement True or False. Justify each answer.

23. a. A homogeneous equation is always consistent.

b. The equation $Ax = 0$ gives an explicit description of its solution set.

c. The homogeneous equation $Ax = 0$ has the trivial solution if and only if the equation has at least one free variable.

d. The equation $x = p + tv$ describes a line through v parallel to p.

e. The solution set of $Ax = b$ is the set of all vectors of the form $w = p + v_h$, where v_h is any solution of the equation $Ax = 0$.

24. a. If x is a nontrivial solution of $Ax = 0$, then every entry in x is nonzero.

b. The equation $x = x_2 u + x_3 v$, with x_2 and x_3 free (and neither u nor v a multiple of the other), describes a plane through the origin.

c. The equation $Ax = b$ is homogeneous if the zero vector is a solution.

d. The effect of adding p to a vector is to move the vector in a direction parallel to p.

e. The solution set of $Ax = b$ is obtained by translating the solution set of $Ax = 0$.

25. Prove Theorem 6:

a. Suppose p is a solution of $Ax = b$, so that $Ap = b$. Let v_h be any solution of the homogeneous equation $Ax = 0$, and let $w = p + v_h$. Show that w is a solution of $Ax = b$.

b. Let w be any solution of $Ax = b$, and define $v_h = w - p$. Show that v_h is a solution of $Ax = 0$. This shows that every solution of $Ax = b$ has the form $w = p + v_h$, with p a particular solution of $Ax = b$ and v_h a solution of $Ax = 0$.

26. Suppose $Ax = b$ has a solution. Explain why the solution is unique precisely when $Ax = 0$ has only the trivial solution.

27. Suppose A is the 3×3 *zero* matrix (with all zero entries). Describe the solution set of the equation $Ax = 0$.

28. If $b \neq 0$, can the solution set of $Ax = b$ be a plane through the origin? Explain.

In Exercises 29–32, (a) does the equation $Ax = 0$ have a nontrivial solution and (b) does the equation $Ax = b$ have at least one solution for every possible b?

29. A is a 3×3 matrix with three pivot positions.

30. A is a 3×3 matrix with two pivot positions.

31. A is a 3×2 matrix with two pivot positions.

32. A is a 2×4 matrix with two pivot positions.

33. Given $A = \begin{bmatrix} -2 & -6 \\ 7 & 21 \\ -3 & -9 \end{bmatrix}$, find one nontrivial solution of $Ax = 0$ by inspection. [*Hint:* Think of the equation $Ax = 0$ written as a vector equation.]

34. Given $A = \begin{bmatrix} 4 & -6 \\ -8 & 12 \\ 6 & -9 \end{bmatrix}$, find one nontrivial solution of $Ax = 0$ by inspection.

35. Construct a 3×3 nonzero matrix A such that the vector $\begin{bmatrix} 1 \\ 1 \\ 1 \end{bmatrix}$ is a solution of $Ax = 0$.

36. Construct a 3×3 nonzero matrix A such that the vector $\begin{bmatrix} 1 \\ -2 \\ 1 \end{bmatrix}$ is a solution of $Ax = 0$.

37. Construct a 2×2 matrix A such that the solution set of the equation $Ax = 0$ is the line in $\mathbb{R}^2$ through $(4, 1)$ and the origin. Then, find a vector b in $\mathbb{R}^2$ such that the solution set of $Ax = b$ is *not* a line in $\mathbb{R}^2$ parallel to the solution set of $Ax = 0$. Why does this *not* contradict Theorem 6?

38. Suppose A is a 3×3 matrix and y is a vector in $\mathbb{R}^3$ such that the equation $Ax = y$ does *not* have a solution. Does there exist a vector z in $\mathbb{R}^3$ such that the equation $Ax = z$ has a unique solution? Discuss.

39. Let A be an $m \times n$ matrix and let u be a vector in $\mathbb{R}^n$ that satisfies the equation $Ax = 0$. Show that for any scalar c, the vector cu also satisfies $Ax = 0$. [That is, show that $A(cu) = 0$.]

40. Let A be an $m \times n$ matrix, and let u and v be vectors in $\mathbb{R}^n$ with the property that $Au = 0$ and $Av = 0$. Explain why $A(u + v)$ must be the zero vector. Then explain why $A(cu + dv) = 0$ for each pair of scalars c and d.