1. Let X be an $n \times 2$ matrix, and suppose that the columns of X are linearly dependent. Show that X^TX is **not** invertible.

ANS: Since the columns of X are linearly dependent, we can assume without loss of generality that X can be written as

$$X = \begin{bmatrix} 1 & a_0 \\ 1 & a_0 \\ \vdots & \vdots \\ 1 & a_0 \end{bmatrix}$$

where a_0 is a real number. Then X^TX equals

$$X^TX = \begin{bmatrix} 1 & 1 & \dots & 1 \\ a_0 & a_0 & \dots & a_0 \end{bmatrix} \begin{bmatrix} 1 & a_0 \\ 1 & a_0 \\ \vdots & \vdots \\ 1 & a_0 \end{bmatrix} = \begin{bmatrix} n & na_0 \\ na_0 & na_0^2 \end{bmatrix}$$

The determinant of the 2×2 matrix is $(n)(na_0^2) - (na_0)(na_0) = 0$. Therefore X^TX is **not** invertible.

For the remaining questions, A is an $m \times n$ matrix.

2. Show that if $A\mathbf{x} = \mathbf{0}$, then $A^TA\mathbf{x} = \mathbf{0}$

ANS:

$$A\mathbf{x} = \mathbf{0} \Rightarrow A^TA\mathbf{x} = A^T\mathbf{0} \Rightarrow A^TA\mathbf{x} = \mathbf{0}$$

3. Show that if $A^TA\mathbf{x} = \mathbf{0}$, then $A\mathbf{x} = \mathbf{0}$. Hint: Use the fact that $\mathbf{x}^TA^TA\mathbf{x} = 0$, and compare with exercise 25, page 108.

ANS: By the (infamous) Theorem 3(d) on page 106, $\mathbf{x}^TA^TA\mathbf{x} = 0$ can be rewritten as $(A\mathbf{x})^T(A\mathbf{x}) = 0$. Now $(A\mathbf{x})^T$ is a matrix of size $1 \times m$ and $A\mathbf{x}$ is a vector of size $m \times 1$. Therefore, their product is of size 1×1, i.e. a real number. Let $\mathbf{u} = A\mathbf{x} = \begin{bmatrix} u_0 \\ u_1 \\ \vdots \\ u_m \end{bmatrix}$. Then $(A\mathbf{x})^T(A\mathbf{x}) = \mathbf{u}^T\mathbf{u} = u_0^2 + u_1^2 + \ldots + u_m^2 = 0$. The product $\mathbf{u}^T\mathbf{u}$ can equal 0 only if $u_0 = u_1 = \ldots = u_m = 0$, i.e. $\mathbf{u} = \mathbf{0}$. Therefore, since $\mathbf{u} = A\mathbf{x}$, we must have $A\mathbf{x} = \mathbf{0}$.

4. Show why problems **2** and **3** imply that the columns of A are linearly independent if and only if the columns of A^TA are linearly independent. Hint: What do problems **2** and **3** say about how the solution set of $A\mathbf{x} = \mathbf{0}$ relates to the solution set of $A^TA\mathbf{x} = \mathbf{0}$?

ANS: In Problem **2** we showed that if $\mathbf{x_0}$ is in the solution set of $A\mathbf{x} = \mathbf{0}$ (i.e. plugging $\mathbf{x_0}$ into the equation yields $A\mathbf{x_0} = \mathbf{0}$), then $\mathbf{x_0}$ is also in the solution set of $A^TA\mathbf{x} = \mathbf{0}$. Therefore, the solution set of $A\mathbf{x} = \mathbf{0}$ is contained in (a subset of) the solution set of $A^TA\mathbf{x} = \mathbf{0}$.

In Problem **3** we showed that if $\mathbf{x_0}$ is in the solution set of $A^TA\mathbf{x} = \mathbf{0}$, then $\mathbf{x_0}$ is also in the solution set of $A\mathbf{x} = \mathbf{0}$. Therefore, the solution set of $A^TA\mathbf{x} = \mathbf{0}$ is a subset of the solution set of $A\mathbf{x} = \mathbf{0}$.

Each solution set can be a subset of the other only if both solution sets are *equal*. Therefore, the columns of A are linearly independent if and only if $A\mathbf{x} = \mathbf{0}$ has *only* the trivial solution, which happens if and only if $A^TA\mathbf{x} = \mathbf{0}$ has *only* the trivial solution, which happens if and only if the columns of A^TA are linearly independent.

5. Suppose the columns of A are linearly independent. Use Problems **2** and **3** to show that A^TA is invertible.

ANS: Suppose the columns of A are linearly independent. By Problems **2** and **3**, we know that the columns of A^TA are linearly independent. This implies that $A^TA\mathbf{x} = \mathbf{0}$ has only the trivial solution, which implies that A^TA has n pivot columns. A^TA has n pivot columns implies that A^TA is row equivalent to the identity matrix I, and this implies that A^TA is invertible.

OR

Suppose the columns of A are linearly independent. By Problems **2** and **3**, we know that the columns of A^TA are linearly independent. Therefore, by the Invertible Matrix Theorem (i.e. the Über Theorem), A^TA is invertible.