Lecture 9: Intro to matrix algebra. Inverses.

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Suppose your data take the form of vectors $\mathbf{x}_k \in \mathbb{R}^n$, where $k = 0, 1, 2, \ldots$s. If there is an $n \times n$ matrix A such that $\mathbf{x}_1 = A\mathbf{x}_0, \mathbf{x}_2 = A\mathbf{x}_1$ and generally $\mathbf{x}_{k+1} = A\mathbf{x}_k$ (*) , we say that equation (*) is a linear difference equation

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where r_0 is the population in the city in year 0 and s_0 is the population in the suburb in year 0. The vectors $\mathbf{x}_1 = \begin{bmatrix} r_1 \\ s_1 \end{bmatrix}, \mathbf{x}_2 = \begin{bmatrix} r_2 \\ s_2 \end{bmatrix}$ record the popluation distribution in year 1, year 2, etc.

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Matrix algebra

Matrix algebra is like regular algebra, except instead of adding and multiplying real numbers, you add and multiply matrices.

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We use this notation because it allows us to define operations on matrices very neatly.

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$$A + B = \begin{bmatrix} 1 & 10 \\ 3 & 5 \\ 4 & 1 \end{bmatrix}.$$

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Row-column multiplication of $A\mathbf{x}$

Fact

Let A be an $m \times n$ matrix and let $\mathbf{x} \in \mathbb{R}^n$. Then

$$A\mathbf{x} = \begin{bmatrix} a_{11}x_1 + a_{12}x_2 + \ldots + a_{1n}x_n \\ a_{21}x_1 + a_{22}x_2 + \ldots + a_{2n}x_n \\ \vdots \\ a_{m1}x_1 + a_{m2}x_2 + \ldots + a_{mn}x_n \end{bmatrix}.$$

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Composing functions

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for all $\mathbf{x} \in \mathbb{R}^n$.

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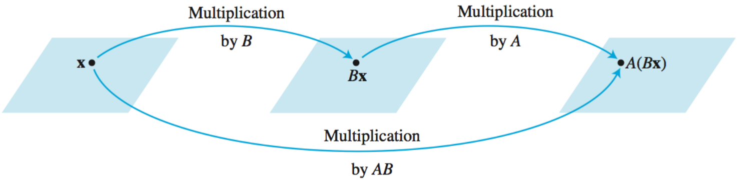
Since multiplying by matrices is equivalent to applying linear transformations, we can define the product matrix to be the the matrix corresponding to the composition of the linear transformations.

$$A(B\mathbf{x}) = (AB)\mathbf{x} \quad (1)$$

See Fig. 3.



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Thus we can compute the kth column of AB by multiplying the kth column of B by A.

Multiplying matrices

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Thus the product is

$$AB = \begin{bmatrix} A\mathbf{b}_1 & A\mathbf{b}_2 \end{bmatrix} = \begin{bmatrix} 11 & 38 \\ 3 & 3 \end{bmatrix}.$$

When can you multiply two matrices?

We have defined the product AB of the as

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Those of you familiar with dot products/inner products will recognize this as the **dot product** of the ith row of A with the jth column of B.

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Example

Let $A = \begin{bmatrix} 15 & -8 & 20 & 30 & 2 \\ -4 & 7 & 13 & 11 & 6 \end{bmatrix}$ and

$$B = \begin{bmatrix} 1 & 2 & 3 & 4 & 5 & 6 & 7 \\ 8 & 9 & 10 & 11 & 12 & 13 & 14 \\ 15 & 16 & 17 & 18 & 19 & 20 & 21 \\ 22 & 23 & 24 & 25 & 26 & 27 & 28 \\ 29 & 30 & 31 & 32 & 33 & 34 & 35 \end{bmatrix}.$$

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Then the product AB is a 2×7 matrix. What is the $(2, 4)$-th entry? Just add up the products of the entries in the 2nd row of A and the 4th column of B.

$$(AB)_{2,4} = (-4)(4)+(7)(11)+(13)(18)+(11)(25)+(6)(32) = 762.$$

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$$\begin{bmatrix} a_{i1} & a_{i2} & \dots & a_{in} \end{bmatrix} \begin{bmatrix} b_{11} & b_{12} & \dots & b_{1p} \\ b_{21} & b_{22} & \dots & b_{2p} \\ \vdots & \vdots & \vdots & \vdots \\ b_{n1} & b_{n2} & \dots & b_{np} \end{bmatrix}$$

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Properties of matrix multiplication

Matrix multiplication "inherits" a lot of the nice properties of matrix-vector products.

Definition

Here and in every possible future lecture, for any integer $m \geq 1$ we denote by I_m the $m \times m$ **identity matrix**

$$I_m = \begin{bmatrix} 1 & 0 & \dots & 0 \\ 0 & 1 & \dots & 0 \\ \vdots & \vdots & \vdots & \vdots \\ 0 & 0 & \dots & 1 \end{bmatrix}$$

which is the unique matrix which satisfies $I_m\mathbf{x} = \mathbf{x}$ for all $\mathbf{x} \in \mathbb{R}^m$.

Properties of matrix multiplication

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$$A^k = \begin{bmatrix} 2^k & 2^k \\ 0 & 0 \end{bmatrix}.$$

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There's a bunch of simple properties of the tranpose operation $A \mapsto A^T$ that we record for posterity's sake.

Theorem

Let A and B denote matrices, where in each case the sums or products are defined as they need to be.

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Most of these are really straightforward computations on entries $[a_{ij}]$. The last one is a little bit of work, and it's worth remembering that in general

$$(AB)^T \neq A^T B^T.$$