Lecture 8: The matrix of a linear transformation. Applications

Danny W. Crytser

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Example

Let $T : \mathbb{R}^2 \to \mathbb{R}^3$ be the linear transformation defined by

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Definition

If $T : \mathbb{R}^n \to \mathbb{R}^m$ is a linear transformation and $\mathbf{e}_1, \mathbf{e}_2, \ldots, \mathbf{e}_n$ are the standard basis vectors in $\mathbb{R}^n$,

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$$A = \begin{bmatrix} T(\mathbf{e}_1) & T(\mathbf{e}_2) & \ldots & T(\mathbf{e}_n) \end{bmatrix}$$

which satisfies $T(\mathbf{x}) = A\mathbf{x}$ for all $\mathbf{x} \in \mathbb{R}^n$ is called the **standard matrix for** T.

There are a few different types of linear transformations $\mathbb{R}^2 \to \mathbb{R}^2$ that we can describe with words ("rotate the plane counterclockwise by $\pi/2$") and then we get the matrix just by tracking the image of the basis vectors $\mathbf{e}_1 = \begin{bmatrix} 1 \\ 0 \end{bmatrix}$ and $\mathbf{e}_2 = \begin{bmatrix} 0 \\ 1 \end{bmatrix}$.

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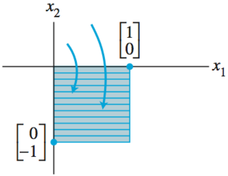
TABLE 1 Reflections

Transformation	Image of the Unit Square	Standard Matrix
Reflection through the x_1-axis		$\begin{bmatrix} 1 & 0 \\ 0 & -1 \end{bmatrix}$

Reflection through $x_1 = x_2$

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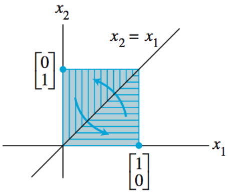
Reflection through the line $x_2 = x_1$



$$\begin{bmatrix} 0 & 1 \\ 1 & 0 \end{bmatrix}$$

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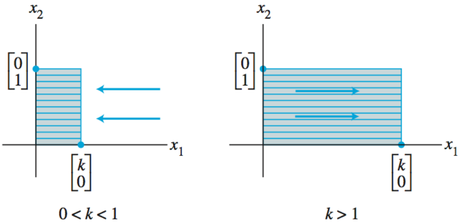
TABLE 2 Contractions and Expansions

Transformation	Image of the Unit Square	Standard Matrix
Horizontal contraction and expansion		$\begin{bmatrix} k & 0 \\ 0 & 1 \end{bmatrix}$

Shear transformations

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TABLE 3 Shears

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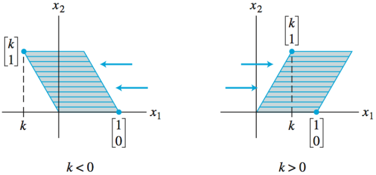
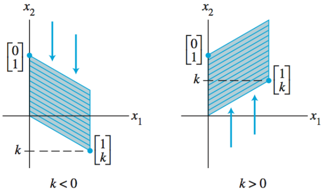
Horizontal shear		$\begin{bmatrix} 1 & k \\ 0 & 1 \end{bmatrix}$
Vertical shear		$\begin{bmatrix} 1 & 0 \\ k & 1 \end{bmatrix}$

Onto

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A function $T : \mathbb{R}^n \to \mathbb{R}^m$ is said to be **onto** if the range is the codomain, that is, for each vector $\mathbf{y} \in \mathbb{R}^m$ there is at least one $\mathbf{x} \in \mathbb{R}^n$ with $T(\mathbf{x}) = \mathbf{y}$.

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Example

Let $T : \mathbb{R}^2 \to \mathbb{R}^2$ be given by $T(x_1, x_2) = (x_1, 0)$. Then T is not onto: the range is the x-axis, an object in mathematics noteworthy for not being the entire plane.

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The quality of being onto has to do with existence of solutions: a linear transformation T given by $T(\mathbf{x}) = A\mathbf{x}$ is onto if $A\mathbf{x} = \mathbf{b}$ is consistent for all $\mathbf{b} \in \mathbb{R}^m$.

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Theorem

Let A be an $m \times n$ matrix. Then $A\mathbf{x} = \mathbf{b}$ is consistent for all $\mathbf{b} \in \mathbb{R}^m$ if and only if every row in the echelon form of A (not augmented) has a nonzero entry.

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You can see that the only way that T can be both onto and one-to-one is if $m = n$.

Example

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Let $A = \begin{bmatrix} 1 & 7 \\ 2 & 3 \\ 4 & 2 \end{bmatrix}$ and define $T : \mathbb{R}^2 \to \mathbb{R}^3$ by $T(\mathbf{x}) = A\mathbf{x}$.

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Let $A = \begin{bmatrix} 1 & 7 \\ 2 & 3 \\ 4 & 2 \end{bmatrix}$ and define $T : \mathbb{R}^2 \to \mathbb{R}^3$ by $T(\mathbf{x}) = A\mathbf{x}$. Is T onto? Is T one-to-one?

APPLICATIONS

Voltage loops

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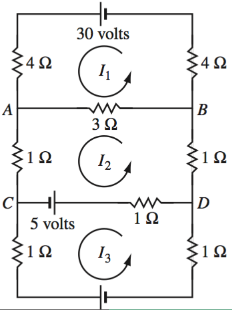

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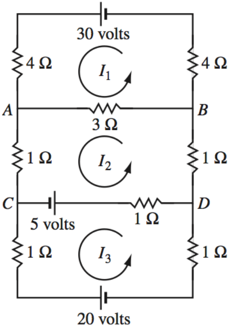

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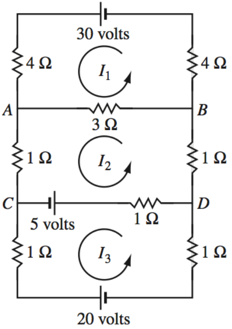

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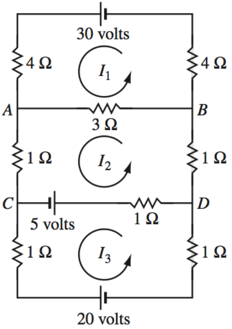

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Passing from A to B the sum of the voltage drops is $3I_1 - 3I_2$. (Notice that the voltage source in the first loop is positive.)

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Remember: when adding voltage sources you have to check to see if they're positive (current runs positive terminal to negative terminal) or negative (vice versa).

Kirchhoff: example

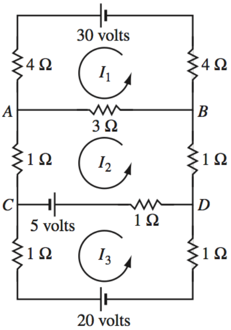

FIGURE 1

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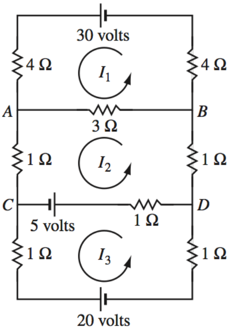

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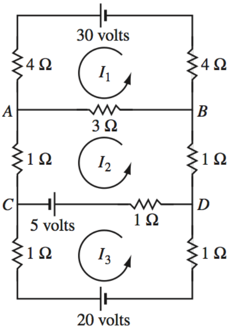

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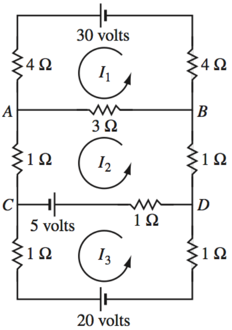

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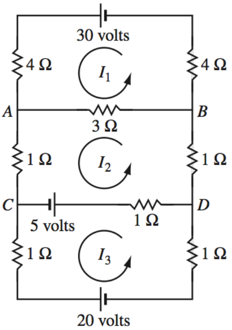

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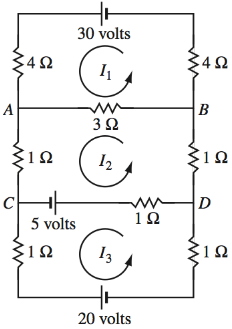

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$$11I_1 - 3I_2 = 30 \qquad (1)$$

$$-3I_1 + 6I_2 - I_3 = 5 \qquad (2)$$

$$-I_2 + 3I_3 = -25 \qquad (3)$$

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Has a unique solution: $I_1 = 3$ amps, $I_2 = 1$ amps, $I_3 = -8$ amps.

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Difference equations

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In many situations you will be measuring some system and all the information about the system at time $k$will be contained in some vector $\mathbf{x}_k$. (Could be age, salary, population, microbe count, whatever).

Definition

Suppose your data take the form of vectors $\mathbf{x}_k \in \mathbb{R}^n$, where $k = 0, 1, 2, \ldots$s. If there is an $n \times n$ matrix A such that $\mathbf{x}_1 = A\mathbf{x}_0, \mathbf{x}_2 = A\mathbf{x}_1$ and generally $\mathbf{x}_{k+1} = A\mathbf{x}_k$ (*) , we say that equation (*) is a linear difference equation (some people call it a recursion relation, because it gives new measurement in terms of the old measurement).

Difference equations and population

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where r_0 is the population in the city in year 0 and s_0 is the population in the suburb in year 0. The vectors $\mathbf{x}_1 = \begin{bmatrix} r_1 \\ s_1 \end{bmatrix}, \mathbf{x}_2 = \begin{bmatrix} r_2 \\ s_2 \end{bmatrix}$ record the popluation distribution in year 1, year 2, etc.

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$$\begin{bmatrix} r_1 \\ s_1 \end{bmatrix} = r_0 \begin{bmatrix} .95 \\ .05 \end{bmatrix} + \begin{bmatrix} .03 \\ .97 \end{bmatrix} = \begin{bmatrix} .95 & .03 \\ .05 & .97 \end{bmatrix}.$$

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If A is the matrix above, then we can write $\mathbf{x}_k = A\mathbf{x}_{k-1}$. You can use this to predict the future.