Math 22: Linear Algebra

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$$a_1 x_1 + \ldots + a_n x_n = c$$

where the $a_1, \ldots, a_n$ are fixed coefficients, the $x_1, \ldots, x_n$ are variables, and c is a constant.

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$$\sqrt{x} + \sin(y) = 1$$

is not a linear equation.

What is a system of linear equations?

A *system of linear equations* is just a collection of linear equations using the same variables,

$$\begin{aligned} a_{11}x_1 + a_{12}x_2 + \ldots + a_{1n}x_n &= c_1 \\ a_{21}x_1 + a_{22}x_2 + \ldots + a_{2n}x_n &= c_2 \\ \vdots \qquad \vdots \qquad & \vdots \\ a_{m1}x_1 + a_{m2}x_2 + \ldots + a_{mn}x_n &= c_m \end{aligned}$$

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Here is a system of linear equations with the variables x, y.

$$\begin{aligned} 2x + 7y &= 1 \\ x + 3y &= 0 \end{aligned}$$

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The solution set is $\{(-3, 1)\}$–no other solutions are possible.

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The book has a nice illustration of these possibilities for the case when there are $n = 2$ variables (parallel, skew, and identical lines).

How do we find the solutions?

In general, our strategy is to replace the given system with an *equivalent system*: a new system of linear equations which has exactly the same solution set.

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Now subtract -2 times the first equation from the second equation:

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$$2x + 7y = 1$$

Now subtract -2 times the first equation from the second equation:

$$x + 3y = 0$$
$$0 + y = 1$$

Now subtract 3 times the second equation from the first row:

$$x + 0 = -3$$
$$0 + y = 1$$

The solution to this system is $(-3, 1)$.

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Matrices

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has two matrices affiliated to it: the coefficient matrix

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and the augmented matrix (including constants)

$$\begin{bmatrix} 2 & 7 & 1 \\ 1 & 3 & 0 \end{bmatrix}$$

Solving systems with matrices

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- **scaling** Multiply every term/entry of an equation/row by a nonzero number $c \neq 0$.

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Let's solve a larger system.

$$
\begin{aligned}
x_1 - 2x_2 + x_3 &= 0 \\
2x_2 - 8x_3 &= 8 \\
-4x_1 + 5x_2 + 9x_3 &= 9
\end{aligned}
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First convert to an augmented matrix.

$$\begin{bmatrix} 1 & -2 & 1 & 0 \\ 0 & 2 & -8 & 8 \\ -4 & 5 & 9 & -9 \end{bmatrix}$$

Add 4 times row 1 to row 3. This eliminates the x_1 variable from the third equation:

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$$\begin{bmatrix} 1 & -2 & 1 & 0 \\ 0 & 2 & -8 & 8 \\ -4 & 5 & 9 & -9 \end{bmatrix} \to \begin{bmatrix} 1 & -2 & 1 & 0 \\ 0 & 2 & -8 & 8 \\ 0 & -3 & 13 & -9 \end{bmatrix}$$

Multiply the second row by $\frac{1}{2}$. This makes the x_2 variable appear with coefficient 1 in the second equation:

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$$\begin{bmatrix} 1 & -2 & 1 & 0 \\ 0 & 2 & -8 & 8 \\ 0 & -3 & 13 & -9 \end{bmatrix} \to \begin{bmatrix} 1 & -2 & 1 & 0 \\ 0 & 1 & -4 & 4 \\ 0 & -3 & 13 & -9 \end{bmatrix}$$

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$$\left[\begin{array}{cccc} 1 & -2 & 1 & 0 \\ 0 & 1 & -4 & 4 \\ 0 & -3 & 13 & -9 \end{array}\right] \rightarrow \left[\begin{array}{cccc} 1 & -2 & 1 & 0 \\ 0 & 1 & -4 & 4 \\ 0 & 0 & 1 & 3 \end{array}\right]$$

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$$\begin{bmatrix} 1 & -2 & 1 & 0 \\ 0 & 1 & -4 & 4 \\ 0 & 0 & 1 & 3 \end{bmatrix} \to \begin{bmatrix} 1 & -2 & 1 & 0 \\ 0 & 1 & 0 & 16 \\ 0 & 0 & 1 & 3 \end{bmatrix}$$

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$$\begin{bmatrix} 1 & -2 & 0 & -3 \\ 0 & 1 & 0 & 16 \\ 0 & 0 & 1 & 3 \end{bmatrix} \to \begin{bmatrix} 1 & 0 & 0 & 29 \\ 0 & 1 & 0 & 16 \\ 0 & 0 & 1 & 3 \end{bmatrix}$$

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This new matrix represents the system

$$\begin{aligned} x_1 &= 29 \\ x_2 &= 16 \\ x_3 &= 3 \end{aligned}$$

This is the solution to our original system (Check).

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The previous example is consistent. The system

$$\begin{aligned} 2x + y &= 0 \\ -4x - 2y &= 1 \end{aligned}$$

is inconsistent. (Add 2 times the first row to the second row: you obtain $0 = 1$.)

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These are the two main questions we will begin with.