## Lecture 13: Vector spaces

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- We will discuss subspaces of ℝ<sup>n</sup>: subsets in which you can add and scale vectors.
- We will talk about bases
- Olumn space and null space of a matrix; finding bases for these spaces.
- Vector spaces
- Subspaces of vector spaces

We have drawn a lot of attention to the fact that for a linear transformation  $T : \mathbb{R}^n \to \mathbb{R}^m$ , the range need not equal the codomain  $(\mathbb{R}^m)$ .

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- **2** If  $\mathbf{u}, \mathbf{v} \in H$ , then  $\mathbf{u} + \mathbf{v} \in H$ . (*Closed under addition*.)
- So For each u ∈ H and each scalar c ∈ ℝ, the vector cu ∈ H (Closed under scalar multiplication.)

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- **③** in  $\mathbb{R}^2$ , if  $H = \{(x, y) \in \mathbb{R}^2 : x, y \text{ are integers}\}$ , then H contains **0** and is closed under addition. Is H a subspace?

## Example

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$$\left(\sum_{i=1}^{p}c_{i}\mathbf{v}_{i}
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and obtain a new linear combination with weights  $c_i + d_i$ . Similarly if k is a scalar

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so we obtain a new linear combination with weights  $kd_i$ . This shows that the span of  $\{\mathbf{v}_1, \ldots, \mathbf{v}_p\}$  is a subspace of  $\mathbb{R}^n$ . Sometimes we'll call Span $\{\mathbf{v}_1, \ldots, \mathbf{v}_p\}$  the *subspace* spanned by  $\{\mathbf{v}_1, \ldots, \mathbf{v}_p\}$ .

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Let A be an  $m \times n$  matrix with columns  $\mathbf{a}_1, \ldots, \mathbf{a}_n$ . Then the **column space** of A is the set  $\operatorname{Col} A := \operatorname{span} \{\mathbf{a}_1, \ldots, \mathbf{a}_n\}$ 

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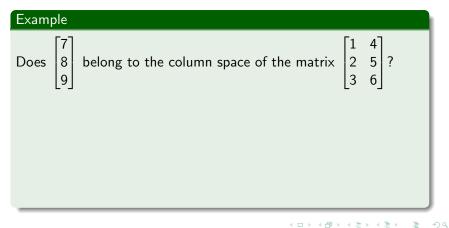
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The vector is a linear combination of the columns of A, so it belongs to the column space of A.

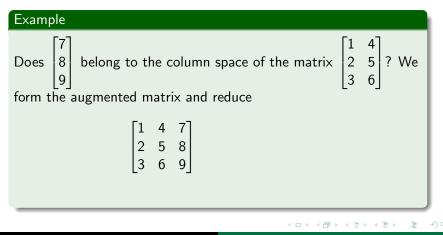
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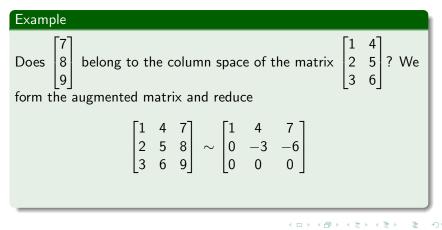
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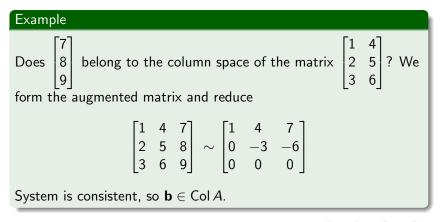
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adjective

- 1 [predic.] having no legal or binding force; invalid: the establishment of a new interim government was declared null and void.
- 2 having or associated with the value zero.
  - $\bullet$  Mathematics (of a set or matrix) having no elements, or only zeros as elements.
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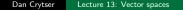
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It's easy to test if **u** belongs to Nul A: just multiply it by A and see if you get  $\mathbf{0}$ .

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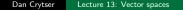
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- (a) if  $A\mathbf{u} = \mathbf{0}$  and c is a scalar, then  $A(c\mathbf{u}) = c(A\mathbf{u}) = c\mathbf{0} = \mathbf{0}$ . So Nul A is closed under scalar multiplication.

## What do you need to span a set?

We've seen that a set  $\mathbf{v}_1, \ldots, \mathbf{v}_p$  of vectors spans a subspace.

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We've seen that a set  $\mathbf{v}_1, \ldots, \mathbf{v}_p$  of vectors spans a subspace. Now we consider how many vectors we actually need to span a subspace.

#### Example

Let 
$$v_1 = (1, 2)$$
 and  $v_2 = (2, 4)$  in  $\mathbb{R}^2$ .

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Let  $\mathbf{v}_1 = (1,2)$  and  $\mathbf{v}_2 = (2,4)$  in  $\mathbb{R}^2$ . Any linear combination of these is just a scalar multiple of  $\mathbf{v}_1$ . Thus

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We are going to be interested in throwing out as many vectors as we can without changing the span.

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#### Remark

It is not enough for a set to be linearly independent in order for it to be a basis, nor is it enough for a set to be spanning. It has to be *both* linearly independent and spanning.

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The columns of any invertible  $n \times n$  matrix form a basis for  $\mathbb{R}^n$ : they are linearly independent and spanning by the Invertible Matrix Theorem. In particular the columns of  $I_n$ , the  $n \times n$  identity matrix, form a basis for  $\mathbb{R}^n$  called the **standard basis** for  $\mathbb{R}^n$ .

$$\mathbf{e}_1 = \begin{bmatrix} 1\\0\\\vdots\\0\\0 \end{bmatrix}, \quad \mathbf{e}_2 = \begin{bmatrix} 0\\1\\0\\\vdots\\0 \end{bmatrix}, \quad \dots, \quad \mathbf{e}_n = \begin{bmatrix} 0\\0\\\vdots\\0\\1 \end{bmatrix}$$

• 
$$S_1 = \{(1,1,0), (2,2,0), (0,0,1)\}$$
 spans  $H$  because any  $(x,x,y) \in H$  is  $(x,x,y) = x(1,1,0) + y(0,0,1) \in \text{span}\{(1,1), (2,2)\}.$ 

- $S_1 = \{(1,1,0), (2,2,0), (0,0,1)\}$  spans *H* because any  $(x,x,y) \in H$  is  $(x,x,y) = x(1,1,0) + y(0,0,1) \in \text{span}\{(1,1), (2,2)\}$ .  $S_1$  is not a basis for *H*. Why?
- $S_2 = \{(1,1,0)\}$  is a linearly independent set in H, because it only has one element and that element is nonzero.

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- S<sub>2</sub> = {(1,1,0)} is a linearly independent set in *H*, because it only has one element and that element is nonzero. However, S<sub>2</sub> is *not* a basis for *H*. Why?

# Finding bases for Nul A

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$$\begin{bmatrix} 1 & 1 & 1 & 0 & 0 \\ 2 & 3 & 1 & 2 & 0 \\ 3 & 2 & 1 & 1 & 0 \end{bmatrix} \sim \begin{bmatrix} 1 & 0 & 0 & 0 & 0 \\ 0 & 1 & 0 & 1 & 0 \\ 0 & 0 & 1 & -1 & 0 \end{bmatrix}$$

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The solution is  $x_1 = 0, x_2 = -x_4, x_3 = x_4, x_4$  free.

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$$\begin{bmatrix} 1 & 1 & 1 & 0 & 0 \\ 2 & 3 & 1 & 2 & 0 \\ 3 & 2 & 1 & 1 & 0 \end{bmatrix} \sim \begin{bmatrix} 1 & 0 & 0 & 0 & 0 \\ 0 & 1 & 0 & 1 & 0 \\ 0 & 0 & 1 & -1 & 0 \end{bmatrix}$$

The solution is  $x_1 = 0, x_2 = -x_4, x_3 = x_4, x_4$  free. Thus

Nul 
$$A = span\{(0, -1, 1, 1)\}$$

and  $\{(0, -1, 1, 1)\}$  is a basis for the null space of A.

Computing a basis for Col A is straightforward:

#### Theorem

Let A be an  $m \times n$  matrix. Then the set of all columns of A which contain pivots form a basis for Col A.

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This requires delicacy to apply: you have to reduce to echelon form to see where the pivots are, but you **do not use the columns of the echelon form**. Computing a basis for Col A is straightforward:

#### Theorem

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This requires delicacy to apply: you have to reduce to echelon form to see where the pivots are, but you **do not use the columns of the echelon form**. You use the columns of the matrix *A*, not the columns of its echelon form.

### Example: basis for Col A

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Let 
$$A = \begin{bmatrix} 1 & 1 & 2 \\ -1 & -1 & 3 \end{bmatrix}$$
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So far we've defined subspaces of  $\mathbb{R}^n$  as things where you can add and scale vectors.

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### Vector spaces: definition

#### Definition

A vector space is a nonempty set V of objects, called vectors, which we can add and multiply by scalars and all the following axioms hold whenever  $\mathbf{u}, \mathbf{v} \in V$ ,  $c, d \in \mathbb{R}$ :

**)** 
$$\mathbf{u} + \mathbf{v} \in V$$

$$\mathbf{3} \ \mathbf{u} + (\mathbf{v} + \mathbf{w}) = (\mathbf{u} + \mathbf{v}) + \mathbf{w}$$

- **③** there is a zero vector  $\mathbf{0} \in V$  such that  $\mathbf{u} + \mathbf{0} = \mathbf{u}$
- **(a)** there is  $-\mathbf{u} \in V$  with  $\mathbf{u} + (-\mathbf{u}) = \mathbf{0}$
- **(**) the scalar multiple  $c\mathbf{u} \in V$

$$c(\mathbf{u} + \mathbf{v}) = c\mathbf{u} + c\mathbf{v}$$

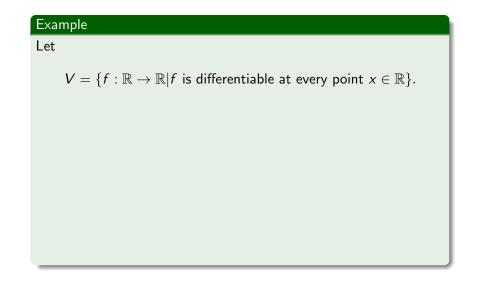
$$(c+d)\mathbf{u} = c\mathbf{u} + d\mathbf{u}$$

$$c(d\mathbf{u}) = (cd)\mathbf{u}$$

# We know that when $V = \mathbb{R}^n$ all these properties of addition and scalar multiplication hold.

We know that when  $V = \mathbb{R}^n$  all these properties of addition and scalar multiplication hold. The idea is that any set with addition and scalar multiplication which plays "this nice" will enjoy all the nice properties of  $\mathbb{R}^n$ .

### Example of vector spaces: differentiable functions



Let

 $V = \{f : \mathbb{R} \to \mathbb{R} | f \text{ is differentiable at every point } x \in \mathbb{R}\}.$ 

We can define pointwise addition and scalar multiplication on this set by

$$(f+g)(x) = f(x) + g(x)$$
$$(cf)(x)c(f(x))$$

for  $f, g \in V$  and  $c \in \mathbb{R}$ . It is a fact from calculus that if f and g are differentiable then f + g is differentiable and cf is differentiable. Thus  $f + g \in V$  and  $cf \in V$ .

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Let m and n be integers and define

$$M = M_{m,n} = \{A = [a_{ij}] : A \text{ is a } m \times n \text{ matrix}\}.$$

We can define addition and scalar multiplication entry-wise the way we discussed in section 2.1.

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#### Remark

Noteice that you get a different vector space for every choice of (m, n): you can only add vectors of the same size.

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#### Remark

Noteice that you get a different vector space for every choice of (m, n): you can only add vectors of the same size. Thus there is the vector space of  $2 \times 2$  matrices, the vector space of  $3 \times 2$  matrices, etc.

### Examples of vector spaces: polynomials

#### Example

Let  $n \ge 1$  be an integer and define

$$\mathbb{P}_n = \{a_0 + a_1t + a_2t^2 + \ldots + a_nt^n : a_0, \ldots, a_n \in \mathbb{R}\}.$$

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Checking that all the vector space axioms hold is kinda boring but within your powers (hah, ugh).

Let V be a vector space and let  $S = {\mathbf{v}_1, \mathbf{v}_2, \dots, \mathbf{v}_p}$  be some subset of vectors in V.



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- if f ∈ H is a function with zero derivative at 0, and c is some scalar, then (cf)'(0) = c(f'(0)) = 0 by the scalar multiple rule for derivatives (or the product rule). Thus f ∈ H and c ∈ ℝ implies cf ∈ H.

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As H satisfies all the three properties, it is a subspace. (a = b)

# A non-example of a subspace: differentiable functions

### Example

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Then H is *not* a subspace

• if f(x) = 1 and  $g(x) = 1 - x^2$ , then  $f, g \in H$  and yet (f+g)(0) = 2, so  $f+g \notin H$ 

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So H violates every one of the three conditions a subset must satisfy in order to be a subspace. Other examples, such as the ones you will encounter on homework, might only violate one or two.

# Example: Upper triangular matrices

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denote the collection of all upper triangular  $3 \times 3$  matrices. Then H is a subspace.

• the zero vector in  $M_3$  is just the zero  $3 \times 3$  matrix, which has no nonzero entries beneath the diagonal and hence belongs to H.

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- If  $A = [a_{ij}] \in H$  and  $c \in \mathbb{R}$ , then  $ca_{ij} = 0$  whenever i > j. So  $cA \in H$  also.

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- or if A = [a<sub>ij</sub>] ∈ H and c ∈ ℝ, then ca<sub>ij</sub> = 0 whenever i > j. So cA ∈ H also. (Closed under scalar multiplication.)

Let  $M_2$  denote the vector space of  $2 \times 2$  matrices. Then

$$H = \{A \in M_2 : \det A \neq 0\},\$$

the set of all invertible  $2 \times 2$  matrices, is *not* a subspace of  $M_2$ .

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Thus *H* is the span of a set of vectors in  $\mathbb{P}_3$ , which means that *H* is automatically a subspace.

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the set of all polynomials with non-negative coefficients, is *not* a subspace of  $\mathbb{P}_3$ . It contains the zero vector because we can set all the coefficients to 0. It is closed under addition because adding non-negative numbers yields non-negative numbers. What's wrong? Why isn't H a subspace?