Lecture 12: Determinants

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Today's lecture

1. We will see several nice properties of determinants that interact meaningfully with matrix operations
2. Learn how row operations alter the determinant of a matrix
3. Learn Cramer's rule which allows us to solve matrix equations by computing determinants

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One of our goals is to determine some tricks for computing the determinant of a matrix.

Determinants and interchange

The elementary row operations each change the determinant in a prescribed way.

Example

Let $A = \begin{bmatrix} 1 & 0 \\ 0 & 1 \end{bmatrix}$. The determinant of A is 1. If we interchange the rows of A, we obtain $E_1 A = \begin{bmatrix} 0 & 1 \\ 1 & 0 \end{bmatrix}$, which has determinant -1.

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Determinants and scaled rows

Example

If $A = \begin{bmatrix} 1 & 0 & 2 \\ 0 & 2 & 7 \\ 0 & 0 & 9 \end{bmatrix}$ then the determinant of A is just the product of the diagonal entries.

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If $A = \begin{bmatrix} 1 & 0 & 2 \\ 0 & 2 & 7 \\ 0 & 0 & 9 \end{bmatrix}$ then the determinant of A is just the product of the diagonal entries. If we scale the second row by 10, to obtain

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Note that if you scale the entire matrix A by the same constant k, that has the same effect as scaling each of the n rows in order: hence $\det(kA) = k^n \det A$.

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Let $A = \begin{bmatrix} 100 & 200 \\ 2 & 3 \end{bmatrix}$. Then $|A| = 100 \begin{vmatrix} 1 & 2 \\ 2 & 3 \end{vmatrix} = -100$.

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So $|A| = 0$.

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Proof.

None of the row replacement operations change the determinant, so we only have to keep track of the effect of the scaling and interchange operations. You can put a matrix in echelon form without any scaling. So the only thing to keep track of is the interchanges. So $|A| = (-1)^r |U|$. As the echelon form is upper triangular, its determinant is the product of its diagonal entries $|U| = u_{11} u_{22} \ldots u_{nn}$ □

Example: determinants and echelon form

Suppose that we have a matrix A and we know that A can be reduced to the echelon form

$$U = \begin{bmatrix} 1 & 2 & 0 \\ 0 & 7 & -1 \\ 0 & 0 & 3 \end{bmatrix}$$

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using some number of replacement operations and one row interchange. What is $|A|$? We use the previous theorem:

$$|A| = (-1)|U| = -(1)(7)(3) = -21.$$

Invertibility and determinant

We see that if A is $n \times n$ and U is an echelon form for A which we obtained using replacements and r interchanges then $|A| = (-1)^r|U| = (-1)^r u_{11} \dots u_{nn}$.

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We see that if A is $n \times n$ and U is an echelon form for A which we obtained using replacements and r interchanges then $|A| = (-1)^r|U| = (-1)^r u_{11} \dots u_{nn}$. A has n pivots if and only if all the entries $u_{11}, \dots, u_{nn}$ are nonzero.

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Fact

This meshes with the 2×2 *case.*

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Compute $\begin{vmatrix} 1 & 2 & 0 & 1 \\ -1 & 3 & 2 & 2 \\ 0 & 5 & 1 & 1 \\ 0 & -5 & 2 & 2 \end{vmatrix}$. First clean out beneath the first pivot

$|A|$

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$|A| = (-5) \begin{vmatrix} 4 & 5 \\ 3 & 3 \end{vmatrix} = (-5)(-3) = 15.$

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Determinants and transpose

Theorem

If A is a square matrix then $\det A = \det A^T$. That is, a matrix and its transpose have the same determinant.

Proof.

You just flip the cofactor expansions: if you compute the determinant of A by cofactor expansion along row j, compute the determinant of A^T along its jth column, and vice versa.

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Any column operation (interchange columns, replace column, scale column) has the same effect on the determinant as the corresponding row operation.

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Proof.

You just flip the cofactor expansions: if you compute the determinant of A by cofactor expansion along row j, compute the determinant of A^T along its jth column, and vice versa. Since you take determinants of transposes of smaller matrices. □

Any column operation (interchange columns, replace column, scale column) has the same effect on the determinant as the corresponding row operation. So replacing columns does not change the determinant, interchanging two columns multiplies the determinant by -1, scaling a column scales the determinant.

Determinants multiply

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Example

We have

$$\underbrace{\begin{bmatrix} 1 & 2 \\ 7 & 5 \end{bmatrix}}_{\det=-9} \underbrace{\begin{bmatrix} 3 & 0 \\ -1 & -1 \end{bmatrix}}_{\det=-3} = \underbrace{\begin{bmatrix} 1 & -2 \\ 16 & -5 \end{bmatrix}}_{\det=27}$$

Non-linearity of the determinant

It is not the case that $\det(A+B)=\det(A)+\det(B)$. For example

$$\underbrace{\begin{bmatrix} 1 & 0 \\ 0 & -1 \end{bmatrix}}_{\det=-1} + \underbrace{\begin{bmatrix} -1 & 0 \\ 0 & 1 \end{bmatrix}}_{\det=-1} = \underbrace{\begin{bmatrix} 0 & 0 \\ 0 & 0 \end{bmatrix}}_{\det=0}$$

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Also it is not true that $\det cA = c \det A$ for all matrices A and scalars c. Instead $\det cA = c^n \det A$.

Cramer's rule

You can use determinants to solve systems $A\mathbf{x} = \mathbf{b}$ for invertible A, without having to find the inverse of A.

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Theorem (Cramer's rule)

Let A be an invertible $n \times n$ matrix. For any $\mathbf{b} \in \mathbb{R}^n$, the unique solution $\mathbf{x} = \begin{bmatrix} x_1 \\ x_2 \\ \vdots \\ x_n \end{bmatrix}$ of $A\mathbf{x} = \mathbf{b}$ has entries given by

$$x_i = \frac{\det A_i(\mathbf{b}}{\det A}$$

Solving a system with Cramer's rule

Example

Let $A = \begin{bmatrix} 1 & 8 \\ 2 & 17 \end{bmatrix}$ (which has determinant 1) and $\mathbf{b} = \begin{bmatrix} 1 \\ 1 \end{bmatrix}$.

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Thus the unique solution is $\mathbf{x} = \begin{bmatrix} 9 \\ -1 \end{bmatrix}$.

Computng areas with determinants

When you are studying geometry you study parallelograms in $\mathbb{R}^2$ and parallelepipeds in $\mathbb{R}^3$. A pair of vectors determines a parallelogram and a triple of vectors determines a parallelepiped.

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Let A be 2×2. If we treat the columns of A as the vertices in a paralleogram that are adjacent to the origin, then the area of that parallelogram is $|\det A|$.

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Theorem

Let A be 2×2. If we treat the columns of A as the vertices in a paralleogram that are adjacent to the origin, then the area of that parallelogram is $|\det A|$. If A is 3×3 and we treat the columns of A as the vertices in a parallelepiped that are adjacent the origin, then the area of that parallelepiped is $|\det A|$.

Examples of areas of parallelograms

Example

Let $A = \begin{bmatrix} 1 & 2 \\ 1 & 0 \end{bmatrix}$. Then the columns of A define a parallelogram with vertices at $(0,0),(1,2),(1,0),(2,2)$.

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Examples of volume of parallelepipeds

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Let $A = \begin{bmatrix} 1 & 3 & 1 \\ 3 & 1 & 1 \\ 0 & 0 & 3 \end{bmatrix}$.

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Let $A = \begin{bmatrix} 1 & 3 & 1 \\ 3 & 1 & 1 \\ 0 & 0 & 3 \end{bmatrix}$. The columns of A define a parallelepiped with vertices $(0,0,0),(1,3,0),(3,1,0),(4,4,0),(1,1,3),(2,4,3),(4,2,3),(5,5,3)$.

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Let $A = \begin{bmatrix} 1 & 3 & 1 \\ 3 & 1 & 1 \\ 0 & 0 & 3 \end{bmatrix}$. The columns of A define a parallelepiped with vertices
$(0,0,0), (1,3,0), (3,1,0), (4,4,0), (1,1,3), (2,4,3), (4,2,3), (5,5,3)$.
The volume of this parallelepiped is

$$|\det A| = 24.$$