# Lecture 11:LU factorizations

## Danny W. Crytser

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- We are going to discuss LU factorizations and how they can be used to solve multiple systems of equations.
- We are going to show how to find LU factorizations for matrices.
- We are going to start discussing determinants.

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(e.g.  $10 = 5 \cdot 2$ ) The same is the case for matrix algebra: we often will take a matrix A and factorize it

$$A = XY$$

for some matrices X and Y with nice properties.

A lot of the time we want to solve a bunch of systems

$$A\mathbf{x}_1 = \mathbf{b}_1 \qquad A\mathbf{x}_2 = \mathbf{b}_2 \qquad \dots$$

where the constant vectors vary but the coefficient matrix stays the same  $% \left( {{{\mathbf{x}}_{i}}} \right)$ 

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where the constant vectors vary but the coefficient matrix stays the same (production in different years with same facilities, etc.) You can speed up your solving with a factorization A = LU.

### Definition

The diagonal entries in an  $m \times n$  matrix  $A = [a_{ij}]$  are  $a_{ii}$ .

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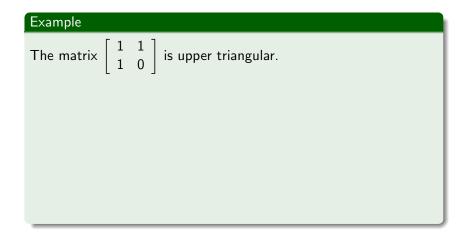
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### Remark

A matrix which is both upper and lower triangular can only have nonzero entries its diagonal, so it is a *diagonal* matrices.



## Example

The matrix 
$$\begin{bmatrix} 1 & 1 \\ 1 & 0 \end{bmatrix}$$
 is upper triangular. The matrix  $\begin{bmatrix} 1 & 0 & 0 \\ 1 & 2 & 0 \\ 7 & 8 & -1 \end{bmatrix}$  is lower triangular (all the entries above the diagonal are zero) but not unit lower triangular.



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# LU factorization

An LU factorization of an  $m \times n$  matrix A looks like



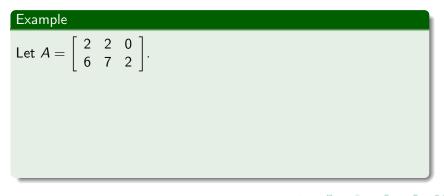
where L is unit Lower triangular and U is Upper triangular.

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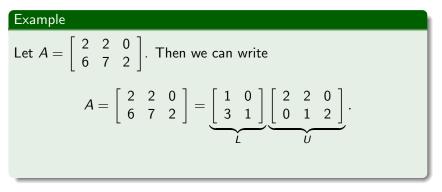
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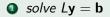
Example Let  $A = \begin{bmatrix} 2 & 2 & 0 \\ 6 & 7 & 2 \end{bmatrix}$ . Then we can write  $A = \begin{bmatrix} 2 & 2 & 0 \\ 6 & 7 & 2 \end{bmatrix} = \underbrace{\begin{bmatrix} 1 & 0 \\ 3 & 1 \end{bmatrix}}_{L} \underbrace{\begin{bmatrix} 2 & 2 & 0 \\ 0 & 1 & 2 \end{bmatrix}}_{U}$ . Note that *L* is square and *U* in this case is not square.

#### Fact

Suppose you have A = LU, where L is unit lower triangular and U is upper triangular. To solve

$$A\mathbf{x} = \mathbf{b}$$

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# Example of solving with LU

Let A be the matrix

$$\begin{bmatrix} 3 & -7 & -2 & 2 \\ -3 & 5 & 1 & 0 \\ 6 & -4 & 0 & -5 \\ -9 & 5 & -5 & 12 \end{bmatrix} = \underbrace{\begin{bmatrix} 1 & 0 & 0 & 0 \\ -1 & 1 & 0 & 0 \\ 2 & -5 & 1 & 0 \\ -3 & 8 & 3 & 1 \end{bmatrix}}_{L} \underbrace{\begin{bmatrix} 3 & -7 & -2 & 2 \\ 0 & -2 & -1 & 2 \\ 0 & 0 & -1 & 1 \\ 0 & 0 & 0 & -1 \end{bmatrix}}_{U}$$

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Solve 
$$A\mathbf{x} = \begin{vmatrix} -9 \\ 5 \\ 7 \\ 1 \end{vmatrix}$$

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Solve 
$$A\mathbf{x} = \begin{bmatrix} -9\\5\\7\\1 \end{bmatrix}$$
 First solve  $L\mathbf{y} = \mathbf{b}$ 
$$\begin{bmatrix} 1 & 0 & 0 & 0 & -9\\-1 & 1 & 0 & 0 & 5\\2 & -5 & 1 & 0 & 7\\-3 & 8 & 3 & 1 & 1 \end{bmatrix}$$

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# Solving ctd

$$\begin{bmatrix} 1 & 0 & 0 & 0 & -9 \\ -1 & 1 & 0 & 0 & 5 \\ 2 & -5 & 1 & 0 & 7 \\ -3 & 8 & 3 & 1 & 1 \end{bmatrix} \sim \begin{bmatrix} 1 & 0 & 0 & 0 & -9 \\ 0 & 1 & 0 & 0 & -4 \\ 0 & 0 & 1 & 0 & 5 \\ 0 & 0 & 0 & 1 & 1 \end{bmatrix}$$
  
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So the solution to  $L\mathbf{y} = \mathbf{b}$  is  $\mathbf{y} = \begin{bmatrix} -9 \\ -4 \\ 5 \\ 1 \end{bmatrix}$ . Now we find the  
solution by solving  $U\mathbf{x} = \mathbf{y}$ , where  $U = \begin{bmatrix} 3 & -7 & -2 & 2 \\ 0 & -2 & -1 & 2 \\ 0 & 0 & -1 & 1 \\ 0 & 0 & 0 & -1 \end{bmatrix}$ .

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$$\begin{bmatrix} 3 & -7 & -2 & 2 & -9 \\ 0 & -2 & -1 & 2 & -4 \\ 0 & 0 & -1 & 1 & 5 \\ 0 & 0 & 0 & -1 & 1 \end{bmatrix}$$
 this is the augmented matrix

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$$\begin{bmatrix} 3 & -7 & -2 & 2 & -9 \\ 0 & -2 & -1 & 2 & -4 \\ 0 & 0 & -1 & 1 & 5 \\ 0 & 0 & 0 & -1 & 1 \end{bmatrix} \sim \begin{bmatrix} 1 & 0 & 0 & 3 \\ 0 & 1 & 0 & 0 & 4 \\ 0 & 0 & 1 & 0 & -6 \\ 0 & 0 & 0 & 1 & -1 \end{bmatrix}$$

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So the solution is  $\mathbf{x} = \begin{bmatrix} 3 \\ 4 \\ -6 \\ -1 \end{bmatrix}$ .

You can check that doing the row reductions of  $L\mathbf{y} = \mathbf{b}$  and  $U\mathbf{x} = \mathbf{y}$  takes a total of 28 operations.

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## Finding the factorization A = LU

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#### Fact

If A is an  $m \times n$  matrix that you can reduce without swapping rows, the following steps factor A = LU:

- First column of L = First column of A scaled to have top entry 1
- 2 Reduce down the first column of A
- Look at the next pivot column, at and below the pivot: that is the next column of L as soon as you divide by the pivot.
- repeat
- If you run out of pivot columns, just use columns from the identity matrix to fill out L.

Let's get the LU factorization for

$$A = \left[ \begin{array}{rrr} 1 & 5 \\ 2 & 6 \\ 3 & 7 \\ 4 & 8 \end{array} \right]$$

*L* will be  $4 \times 4$  and *U* will be  $4 \times 2$ . The first column of *L* is the first column of *A* (we don't need to scale it because the first entry is 1).

$$L = \begin{bmatrix} 1 & 0 & 0 & 0 \\ 2 & * & 0 & 0 \\ 3 & * & * & 0 \\ 4 & * & * & * \end{bmatrix}$$

To get the next column of L we keep reducing

$$\begin{bmatrix} 1 & 5 \\ 2 & 6 \\ 3 & 7 \\ 4 & 8 \end{bmatrix} \sim \begin{bmatrix} 1 & 5 \\ 0 & -4 \\ 0 & -8 \\ 0 & -12 \end{bmatrix}$$

The new pivot is -4. The column at and below the pivot is  $\begin{bmatrix} -4 \\ -8 \\ -12 \end{bmatrix}$ which we scale by  $-\frac{1}{4}$  to obtain the second column of *L*:  $\begin{bmatrix} 1 \\ 2 \\ 3 \end{bmatrix}$ 

Now we have

$$L = \begin{bmatrix} 1 & 0 & 0 & 0 \\ 2 & 1 & 0 & 0 \\ 3 & 2 & * & 0 \\ 4 & 3 & * & * \end{bmatrix}.$$

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Now we have

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$$L = \begin{bmatrix} 1 & 0 & 0 & 0 \\ 2 & 1 & 0 & 0 \\ 3 & 2 & 1 & 0 \\ 4 & 3 & 0 & 1 \end{bmatrix}$$

The matrix U is just the echelon form from the previous slide

$$U = \left[ \begin{array}{rrr} 1 & 5 \\ 0 & -4 \\ 0 & -8 \\ 0 & -12 \end{array} \right]$$

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### Remember the theorem

### Theorem

Let A be a 2 × 2 matrix 
$$A = \begin{bmatrix} a & b \\ c & d \end{bmatrix}$$
. Then A is invertible if and  
only if  $ad - bc \neq 0$ , and  $A^{-1} = \frac{1}{ad - bc} \begin{bmatrix} d & -b \\ -c & a \end{bmatrix}$ .

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when a 2 × 2 matrix is invertible.

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Let 
$$A = \begin{bmatrix} a_{11} & a_{12} & a_{13} \\ a_{21} & a_{22} & a_{23} \\ a_{31} & a_{32} & a_{33} \end{bmatrix}$$
. Let's row reduce:  
$$\begin{bmatrix} a_{11} & a_{12} & a_{13} \\ a_{21} & a_{22} & a_{23} \\ a_{31} & a_{32} & a_{133} \end{bmatrix} \sim \begin{bmatrix} a_{11} & a_{12} & a_{13} \\ 0 & a_{11}a_{22} - a_{12}a_{21} & a_{11}a_{23} - a_{13}a_{21} \\ 0 & a_{11}a_{32} - a_{12}a_{31} & a_{11}a_{33} - a_{13}a_{31} \end{bmatrix}$$

Multiply row 3 by  $a_{11}a_{22} - a_{12}a_{21}$  and then add  $-(a_{11}a_{32} - a_{12}a_{31})$  times row 2.

$$A \sim \begin{bmatrix} a_{11} & a_{12} & a_{13} \\ 0 & a_{11}a_{22} - a_{12}a_{21} & a_{11}a_{23} - a_{13}a_{21} \\ 0 & 0 & a_{11}\Delta \end{bmatrix}$$

where

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where

$$\Delta = a_{11}a_{22}a_{33} + a_{12}a_{23}a_{31} + a_{13}a_{21}a_{32} - a_{11}a_{23}a_{32} - a_{12}a_{21}a_{33} - a_{13}a_{22}a_{31}$$

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We call  $\Delta$  the **determinant** of *A*, denoted det *A*.

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We call  $\Delta$  the **determinant** of *A*, denoted det *A*. In order for *A* to be invertible, it must have a pivot in every column. So det *A* must be nonzero if *A* is to be invertible. We shall see that the converse is true: *A* is invertible if the determinant of *A* is nonzero.

We can write

 $\det A = a_{11}a_{22}a_{33} + a_{12}a_{23}a_{31} + a_{13}a_{21}a_{32} - a_{11}a_{23}a_{32} - a_{12}a_{21}a_{33} - a_{13}a_{22}a_{31}$ as

$$\det A = a_{11} \det \begin{bmatrix} a_{22} & a_{23} \\ a_{32} & a_{33} \end{bmatrix} - a_{12} \det \begin{bmatrix} a_{21} & a_{23} \\ a_{31} & a_{33} \end{bmatrix} + a_{13} \det \begin{bmatrix} a_{21} & a_{22} \\ a_{31} & a_{32} \end{bmatrix}$$

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We can write

$$\det A = a_{11} \det A_{11} - a_{12} \det A_{12} + a_{13} \det A_{13}$$

where  $A_{ij}$  is the matrix you get when you delete row *i* and column *j* from *A*.

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#### Definition

For  $n \ge 2$ , the **determinant** of an  $n \times n$  matrix  $A = [a_{ij}]$  is the sum of *n* terms of the form  $\pm a_{1j} \det A_{1j}$  with plus and minus signs alternating, where the entries  $a_{11}, a_{12}, \ldots, a_{n1}$  are the first row of *A*.

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$$\det A = a_{11} \det A_{11} - a_{12} \det A_{12} + \ldots + (-1)^{1+n} a_{1n} \det A_{1n} \quad (1)$$
$$= \sum_{j=1}^{n} (-1)^{1+j} a_{1j} \det A_{1j} \qquad (2)$$

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We will shortly see more convenient ways of computing the determinant than "across the first row" as in the definition.  $\mathbf{x}$ 

## Computing determinants

Lets compute the determinant of

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$$A = \left[ \begin{array}{rrrr} 1 & 1 & 2 \\ 5 & -1 & 2 \\ 0 & 7 & 3 \end{array} \right]$$

The definition gives

 $\det A =$ 

$$A = \begin{bmatrix} 1 & 1 & 2 \\ 5 & -1 & 2 \\ 0 & 7 & 3 \end{bmatrix}$$

$$\det A = (1) \det \left[ egin{array}{cc} -1 & 2 \ 7 & 3 \end{array} 
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$$\det A = (1) \det \begin{bmatrix} -1 & 2 \\ 7 & 3 \end{bmatrix} + (-1)(1) \det \begin{bmatrix} 5 & 2 \\ 0 & 3 \end{bmatrix}$$

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If A is a matrix then the (i, j)-cofactor of A is the number

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where  $A_{ij}$  is the matrix obtained by deleting the *i*th row and *j*th column from A.

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This is called the **cofactor expansion across the first row of** A.

# Theorem Let A be $n \times n$ .

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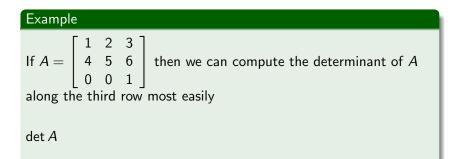
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These are called the cofactor expansions along row i and column j, respectively.



If 
$$A = \begin{bmatrix} 1 & 2 & 3 \\ 4 & 5 & 6 \\ 0 & 0 & 1 \end{bmatrix}$$
 then we can compute the determinant of  $A$  along the third row most easily

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Assume A is upper triangular and take cofactor expansion along first column. Then

$$\det A = a_{11}C_{11} + a_{21}C_{21} + \ldots + a_{n1}C_{n1} = a_{11}C_{11}$$

because all the entries  $a_{k1}$  are zero if k > 1.

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Let

$$A = \begin{bmatrix} 1 & 2 & 3 & 4 \\ 0 & 3 & 4 & 5 \\ 0 & 0 & 2 & 7 \\ 0 & 0 & 0 & -2 \end{bmatrix}$$

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Compute det A.

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Compute det A. We just take the product of the diagonal entries of A: det A = (1)(3)(2)(-2) = -12.

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Remark

WARNING: this only works for triangular matrices.

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Compute $\begin{vmatrix} 5 & -7 & 2 & 2 \\ 0 & 3 & 0 & -4 \\ -5 & -8 & 0 & 3 \\ 0 & 5 & 0 & -6 \end{vmatrix}$	Example									
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along the third column, which only has the (1,3)-entry nonzero:  
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$$= (2)(-1)^{2+1}(-5) \begin{vmatrix} 3 & -4 \\ 5 & -6 \end{vmatrix}$$

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