Lecture 11:LU factorizations

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Today's lecture

1. We are going to discuss LU factorizations and how they can be used to solve multiple systems of equations.
2. We are going to show how to find LU factorizations for matrices.
3. We are going to start discussing determinants.

Factorization

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(e.g. $10 = 5 \cdot 2$) The same is the case for matrix algebra: we often will take a matrix A and factorize it

$$A = XY$$

for some matrices X and Y with nice properties.

Why factorize?

A lot of the time we want to solve a bunch of systems

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$$A\mathbf{x}_1 = \mathbf{b}_1 \qquad A\mathbf{x}_2 = \mathbf{b}_2 \qquad \ldots$$

where the constant vectors vary but the coefficient matrix stays the same (production in different years with same facilities, etc.) You can speed up your solving with a factorization $A = LU$.

Lower and upper triangular matrices

Just as you factor an integer into prime numbers, you choose the kind of matrices into which you factor a matrix.

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The diagonal entries in an $m \times n$ matrix $A = [a_{ij}]$ are a_{ii}.

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Remark

A matrix which is both upper and lower triangular can only have nonzero entries its diagonal, so it is a *diagonal* matrices.

Example

The matrix $\begin{bmatrix} 1 & 1 \\ 1 & 0 \end{bmatrix}$ is upper triangular.

Examples of triangular matrices

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The matrix $\begin{bmatrix} 1 & 1 \\ 1 & 0 \end{bmatrix}$ is upper triangular. The matrix $\begin{bmatrix} 1 & 0 & 0 \\ 1 & 2 & 0 \\ 7 & 8 & -1 \end{bmatrix}$ is lower triangular (all the entries above the diagonal are zero) but not unit lower triangular.

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LU factorization

An LU factorization of an $m \times n$ matrix A looks like

$$A = \underbrace{L}_{m\times m}\underbrace{U}_{m\times n}$$

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Let $A = \begin{bmatrix} 2 & 2 & 0 \\ 6 & 7 & 2 \end{bmatrix}$. Then we can write

$$A = \begin{bmatrix} 2 & 2 & 0 \\ 6 & 7 & 2 \end{bmatrix} = \underbrace{\begin{bmatrix} 1 & 0 \\ 3 & 1 \end{bmatrix}}_{L}\underbrace{\begin{bmatrix} 2 & 2 & 0 \\ 0 & 1 & 2 \end{bmatrix}}_{U}.$$

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Note that L is square and U in this case is not square.

What is an LU factorization good for?

LU factorizations are useful for solving systems via the following two step process.

Fact

Suppose you have $A = LU$, where L is unit lower triangular and U is upper triangular. To solve

$$A\mathbf{x} = \mathbf{b}$$

do the following

1. *solve* $L\mathbf{y} = \mathbf{b}$

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Example of solving with LU

Let A be the matrix

$$\begin{bmatrix} 3 & -7 & -2 & 2 \\ -3 & 5 & 1 & 0 \\ 6 & -4 & 0 & -5 \\ -9 & 5 & -5 & 12 \end{bmatrix} = \underbrace{\begin{bmatrix} 1 & 0 & 0 & 0 \\ -1 & 1 & 0 & 0 \\ 2 & -5 & 1 & 0 \\ -3 & 8 & 3 & 1 \end{bmatrix}}_{L} \underbrace{\begin{bmatrix} 3 & -7 & -2 & 2 \\ 0 & -2 & -1 & 2 \\ 0 & 0 & -1 & 1 \\ 0 & 0 & 0 & -1 \end{bmatrix}}_{U}$$

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Solve $A\mathbf{x} = \begin{bmatrix} -9 \\ 5 \\ 7 \\ 1 \end{bmatrix}$

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Solve $A\mathbf{x} = \begin{bmatrix} -9 \\ 5 \\ 7 \\ 1 \end{bmatrix}$ First solve $L\mathbf{y} = \mathbf{b}$

$$\begin{bmatrix} 1 & 0 & 0 & 0 & -9 \\ -1 & 1 & 0 & 0 & 5 \\ 2 & -5 & 1 & 0 & 7 \\ -3 & 8 & 3 & 1 & 1 \end{bmatrix}.$$

Solving ctd

$$\begin{bmatrix} 1 & 0 & 0 & 0 & -9 \\ -1 & 1 & 0 & 0 & 5 \\ 2 & -5 & 1 & 0 & 7 \\ -3 & 8 & 3 & 1 & 1 \end{bmatrix} \sim \begin{bmatrix} 1 & 0 & 0 & 0 & -9 \\ 0 & 1 & 0 & 0 & -4 \\ 0 & 0 & 1 & 0 & 5 \\ 0 & 0 & 0 & 1 & 1 \end{bmatrix}$$

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$\begin{bmatrix} 3 & -7 & -2 & 2 & -9 \\ 0 & -2 & -1 & 2 & -4 \\ 0 & 0 & -1 & 1 & 5 \\ 0 & 0 & 0 & -1 & 1 \end{bmatrix}$ this is the augmented matrix

Solving ctd ctd

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Row reduction of this augmented matrix yields

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So the solution is $\mathbf{x} = \begin{bmatrix} 3 \\ 4 \\ -6 \\ -1 \end{bmatrix}$.

Why did we use $A = LU$?

You can check that doing the row reductions of $L\mathbf{y} = \mathbf{b}$ and $U\mathbf{x} = \mathbf{y}$ takes a total of 28 operations.

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Fact

If A is an $m \times n$ matrix that you can reduce without swapping rows, the following steps factor $A = LU$:

1. *First column of L = First column of A scaled to have top entry 1*
2. *Reduce down the first column of A*
3. *Look at the next pivot column, at and below the pivot: that is the next column of L as soon as you divide by the pivot.*
4. *repeat*
5. *If you run out of pivot columns, just use columns from the identity matrix to fill out L.*

Example: factorization

Let's get the LU factorization for

$$A = \begin{bmatrix} 1 & 5 \\ 2 & 6 \\ 3 & 7 \\ 4 & 8 \end{bmatrix}$$

L will be 4×4 and U will be 4×2. The first column of L is the first column of A (we don't need to scale it because the first entry is 1).

$$L = \begin{bmatrix} 1 & 0 & 0 & 0 \\ 2 & * & 0 & 0 \\ 3 & * & * & 0 \\ 4 & * & * & * \end{bmatrix}$$

Example: ctd.

To get the next column of L we keep reducing

$$\begin{bmatrix} 1 & 5 \\ 2 & 6 \\ 3 & 7 \\ 4 & 8 \end{bmatrix} \sim \begin{bmatrix} 1 & 5 \\ 0 & -4 \\ 0 & -8 \\ 0 & -12 \end{bmatrix}$$

The new pivot is -4. The column at and below the pivot is $\begin{bmatrix} -4 \\ -8 \\ -12 \end{bmatrix}$ which we scale by $-\frac{1}{4}$ to obtain the second column of L:
$\begin{bmatrix} 1 \\ 2 \\ 3 \end{bmatrix}$

Example: ctd. ctd.

Now we have

$$L = \begin{bmatrix} 1 & 0 & 0 & 0 \\ 2 & 1 & 0 & 0 \\ 3 & 2 & * & 0 \\ 4 & 3 & * & * \end{bmatrix}.$$

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Now there are no more pivots left in A. The rest of the stars we fill up with columns from the appropriate identity matrix.

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$$L = \begin{bmatrix} 1 & 0 & 0 & 0 \\ 2 & 1 & 0 & 0 \\ 3 & 2 & 1 & 0 \\ 4 & 3 & 0 & 1 \end{bmatrix}.$$

The matrix U is just the echelon form from the previous slide

$$U = \begin{bmatrix} 1 & 5 \\ 0 & -4 \\ 0 & -8 \\ 0 & -12 \end{bmatrix}$$

Determinants

Remember the theorem

Theorem

Let A be a 2×2 matrix $A = \begin{bmatrix} a & b \\ c & d \end{bmatrix}$. Then A is invertible if and only if $ad - bc \neq 0$, and $A^{-1} = \frac{1}{ad-bc} \begin{bmatrix} d & -b \\ -c & a \end{bmatrix}$.

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We called $ad - bc$ the **determinant** of $\begin{bmatrix} a & b \\ c & d \end{bmatrix}$. It determines when a 2×2 matrix is invertible.

Determinants for 3×3 matrices

Let $A = \begin{bmatrix} a_{11} & a_{12} & a_{13} \\ a_{21} & a_{22} & a_{23} \\ a_{31} & a_{32} & a_{33} \end{bmatrix}$. Let's row reduce:

$$\begin{bmatrix} a_{11} & a_{12} & a_{13} \\ a_{21} & a_{22} & a_{23} \\ a_{31} & a_{32} & a_{133} \end{bmatrix} \sim \begin{bmatrix} a_{11} & a_{12} & a_{13} \\ 0 & a_{11}a_{22} - a_{12}a_{21} & a_{11}a_{23} - a_{13}a_{21} \\ 0 & a_{11}a_{32} - a_{12}a_{31} & a_{11}a_{33} - a_{13}a_{31} \end{bmatrix}$$

Multiply row 3 by $a_{11}a_{22} - a_{12}a_{21}$ and then add $-(a_{11}a_{32} - a_{12}a_{31})$ times row 2.

Determinants for 3×3 matrices

We row reduced

$$A \sim \begin{bmatrix} a_{11} & a_{12} & a_{13} \\ 0 & a_{11}a_{22} - a_{12}a_{21} & a_{11}a_{23} - a_{13}a_{21} \\ 0 & 0 & a_{11}\Delta \end{bmatrix}$$

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where

$$\Delta = a_{11}a_{22}a_{33} + a_{12}a_{23}a_{31} + a_{13}a_{21}a_{32} - a_{11}a_{23}a_{32} - a_{12}a_{21}a_{33} - a_{13}a_{22}a_{31}$$

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We call Δ the **determinant** of A, denoted $\det A$. In order for A to be invertible, it must have a pivot in every column. So $\det A$ must be nonzero if A is to be invertible. We shall see that the converse is true: A is invertible if the determinant of A is nonzero.

We can write

$$\det A = a_{11}a_{22}a_{33}+a_{12}a_{23}a_{31}+a_{13}a_{21}a_{32}-a_{11}a_{23}a_{32}-a_{12}a_{21}a_{33}-a_{13}a_{22}a_{31}$$

as

$$\det A = a_{11}\det\begin{bmatrix} a_{22} & a_{23} \\ a_{32} & a_{33} \end{bmatrix} - a_{12}\det\begin{bmatrix} a_{21} & a_{23} \\ a_{31} & a_{33} \end{bmatrix} + a_{13}\det\begin{bmatrix} a_{21} & a_{22} \\ a_{31} & a_{32} \end{bmatrix}$$

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We can write

$$\det A = a_{11}\det A_{11} - a_{12}\det A_{12} + a_{13}\det A_{13}$$

where A_{ij} is the matrix you get when you delete row i and column j from A.

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$$\det A = a_{11} \det A_{11} - a_{12} \det A_{12} + \ldots + (-1)^{1+n} a_{1n} \det A_{1n} \quad (1)$$

$$= \sum_{j=1}^{n} (-1)^{1+j} a_{1j} \det A_{1j} \quad (2)$$

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We will shortly see more convenient ways of computing the determinant than "across the first row" as in the definition.

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If A is a matrix then the (i,j)-**cofactor of** A is the number

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This is called the **cofactor expansion across the first row of** A.

Cofactor expansions

Cofactors let us compute the determinant in a lot of different ways, some of which are much more useful than others.

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These are called the cofactor expansions along row i and column j, respectively.

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Example

If $A = \begin{bmatrix} 1 & 2 & 3 \\ 4 & 5 & 6 \\ 0 & 0 & 1 \end{bmatrix}$ then we can compute the determinant of A along the third row most easily

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Assume A is upper triangular and take cofactor expansion along first column. Then

$$\det A = a_{11}C_{11} + a_{21}C_{21} + \ldots + a_{n1}C_{n1} = a_{11}C_{1}1$$

because all the entries a_{k1} are zero if $k > 1$.

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because all the entries a_{k1} are zero if $k > 1$. So $\det A = a_{11} \det A_{11}$ but A_{11} is just a smaller triangular matrix with the diagonal entries $a_{22}, \ldots, a_{nn}$. So $\det A_{11} = a_{22}a_{33} \ldots a_{nn}$.

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WARNING: this only works for triangular matrices. The determinant of $\begin{bmatrix} 1 & 1 \\ 1 & 1 \end{bmatrix}$ is 0, not 1

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