Lecture 10: Invertible matrices. Finding the inverse of a matrix

Danny W. Crytser

April 11, 2014

Today's lecture

Today we will

Today we will

1. Single out a class of especially nice matrices called the *invertible matrices*.

Today's lecture

Today we will

1. Single out a class of especially nice matrices called the *invertible matrices*.
2. Discuss how to compute the inverse of a invertible matrix.

Today's lecture

Today we will

1. Single out a class of especially nice matrices called the *invertible matrices*.
2. Discuss how to compute the inverse of a invertible matrix.
3. State a theorem which says when a square matrix possesses an inverse.

Today's lecture

Today we will

1. Single out a class of especially nice matrices called the *invertible matrices*.
2. Discuss how to compute the inverse of a invertible matrix.
3. State a theorem which says when a square matrix possesses an inverse.

Multiplication of matrices

We saw on Wednesday's lecture that if A is an $m \times n$ matrix and B is an $n \times p$ matrix, then the product AB is defined and

$$AB = \left[\begin{array}{cccc} A\mathbf{b}_1 & A\mathbf{b}_2 & \dots & A\mathbf{b}_p \end{array} \right].$$

Multiplication of matrices

We saw on Wednesday's lecture that if A is an $m \times n$ matrix and B is an $n \times p$ matrix, then the product AB is defined and

$$AB = \begin{bmatrix} A\mathbf{b}_1 & A\mathbf{b}_2 & \dots & A\mathbf{b}_p \end{bmatrix}.$$

Unlike the situation with real numbers, you can find (with not too much difficulty) matrices A, B, C such that

$$AB = AC \text{ and yet } B \neq C.$$

Multiplication of matrices

We saw on Wednesday's lecture that if A is an $m \times n$ matrix and B is an $n \times p$ matrix, then the product AB is defined and

$$AB = \left[\begin{array}{cccc} A\mathbf{b}_1 & A\mathbf{b}_2 & \dots & A\mathbf{b}_p \end{array} \right].$$

Unlike the situation with real numbers, you can find (with not too much difficulty) matrices A, B, C such that

$$AB = AC \text{ and yet } B \neq C.$$

That is, you cannot "divide out by a general matrix A."

Multiplication of matrices

We saw on Wednesday's lecture that if A is an $m \times n$ matrix and B is an $n \times p$ matrix, then the product AB is defined and

$$AB = \left[\begin{array}{cccc} A\mathbf{b}_1 & A\mathbf{b}_2 & \dots & A\mathbf{b}_p \end{array} \right].$$

Unlike the situation with real numbers, you can find (with not too much difficulty) matrices A, B, C such that

$$AB = AC \text{ and yet } B \neq C.$$

That is, you cannot "divide out by a general matrix A." Today we will describe all the matrices that you *can* divide out by.

Multiplication of matrices

We saw on Wednesday's lecture that if A is an $m \times n$ matrix and B is an $n \times p$ matrix, then the product AB is defined and

$$AB = \begin{bmatrix} A\mathbf{b}_1 & A\mathbf{b}_2 & \dots & A\mathbf{b}_p \end{bmatrix}.$$

Unlike the situation with real numbers, you can find (with not too much difficulty) matrices A, B, C such that

$$AB = AC \text{ and yet } B \neq C.$$

That is, you cannot "divide out by a general matrix A." Today we will describe all the matrices that you *can* divide out by. This entails finding the "reciprocal" of a matrix A, which is only possible for some matrices.

The Identity Matrix

We aim to define the inverse of a matrix in (very) rough analogy with the reciprocal of a real number.

The Identity Matrix

We aim to define the inverse of a matrix in (very) rough analogy with the reciprocal of a real number. As $x^{-1} := \frac{1}{x}$, we will need to know what "1" means in a matrix context.

The Identity Matrix

We aim to define the inverse of a matrix in (very) rough analogy with the reciprocal of a real number. As $x^{-1} := \frac{1}{x}$, we will need to know what "1" means in a matrix context.

Definition

Let $m \geq 1$ be an integer.

We aim to define the inverse of a matrix in (very) rough analogy with the reciprocal of a real number. As $x^{-1} := \frac{1}{x}$, we will need to know what "1" means in a matrix context.

Definition

Let $m \geq 1$ be an integer. Then I_m, the $m \times m$ **identity matrix**, is the $m \times m$ matrix given by

$$I_m := \begin{bmatrix} 1 & 0 & \dots & 0 \\ 0 & 1 & \dots & 0 \\ \vdots & \vdots & \vdots & \vdots \\ 0 & 0 & \dots & 1 \end{bmatrix}.$$

The Identity Matrix

We aim to define the inverse of a matrix in (very) rough analogy with the reciprocal of a real number. As $x^{-1} := \frac{1}{x}$, we will need to know what "1" means in a matrix context.

Definition

Let $m \geq 1$ be an integer. Then I_m, the $m \times m$ **identity matrix**, is the $m \times m$ matrix given by

$$I_m := \begin{bmatrix} 1 & 0 & \dots & 0 \\ 0 & 1 & \dots & 0 \\ \vdots & \vdots & \vdots & \vdots \\ 0 & 0 & \dots & 1 \end{bmatrix}.$$

That is, I_m is a diagonal matrix with 1s on the diagonal.

Examples of identity matrices

Example

$$I_2 = \begin{bmatrix} 1 & 0 \\ 0 & 1 \end{bmatrix}$$

Examples of identity matrices

Example

$$I_2 = \begin{bmatrix} 1 & 0 \\ 0 & 1 \end{bmatrix} \qquad I_3 = \begin{bmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{bmatrix}$$

Examples of identity matrices

Example

$$I_2 = \begin{bmatrix} 1 & 0 \\ 0 & 1 \end{bmatrix} \qquad I_3 = \begin{bmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{bmatrix} \qquad I_4 = \begin{bmatrix} 1 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 \\ 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 1 \end{bmatrix}$$

Examples of identity matrices

Example

$$I_2 = \begin{bmatrix} 1 & 0 \\ 0 & 1 \end{bmatrix} \qquad I_3 = \begin{bmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{bmatrix} \qquad I_4 = \begin{bmatrix} 1 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 \\ 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 1 \end{bmatrix}$$

The interesting thing about the identity matrices is that for any $m \times n$ matrix A, $I_m A = A$ and $A I_n = A$.

Examples of identity matrices

Example

$$I_2 = \begin{bmatrix} 1 & 0 \\ 0 & 1 \end{bmatrix} \qquad I_3 = \begin{bmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{bmatrix} \qquad I_4 = \begin{bmatrix} 1 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 \\ 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 1 \end{bmatrix}$$

The interesting thing about the identity matrices is that for any $m \times n$ matrix A, $I_m A = A$ and $AI_n = A$. (Use row-column rule).

Examples of identity matrices

Example

$$I_2 = \begin{bmatrix} 1 & 0 \\ 0 & 1 \end{bmatrix} \qquad I_3 = \begin{bmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{bmatrix} \qquad I_4 = \begin{bmatrix} 1 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 \\ 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 1 \end{bmatrix}$$

The interesting thing about the identity matrices is that for any $m \times n$ matrix A, $I_m A = A$ and $AI_n = A$. (Use row-column rule). Sometimes when the context is abundantly clear, we'll write I instead of I_n, but we'll try to retain the subscript to avoid confusion.

Examples of identity matrices

Example

$$I_2 = \begin{bmatrix} 1 & 0 \\ 0 & 1 \end{bmatrix} \qquad I_3 = \begin{bmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{bmatrix} \qquad I_4 = \begin{bmatrix} 1 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 \\ 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 1 \end{bmatrix}$$

The interesting thing about the identity matrices is that for any $m \times n$ matrix A, $I_m A = A$ and $AI_n = A$. (Use row-column rule). Sometimes when the context is abundantly clear, we'll write I instead of I_n, but we'll try to retain the subscript to avoid confusion.

Invertible matrices

Definition

An $n \times n$ matrix A is **invertible** if there exists an $n \times n$ matrix C such that

$$AC = I_n = CA.$$

Definition

An $n \times n$ matrix A is **invertible** if there exists an $n \times n$ matrix C such that

$$AC = I_n = CA.$$

If such a C exists, we denote it by A^{-1} and refer to it as the **inverse** of A.

Invertible matrices

Definition

An $n \times n$ matrix A is **invertible** if there exists an $n \times n$ matrix C such that

$$AC = I_n = CA.$$

If such a C exists, we denote it by A^{-1} and refer to it as the **inverse** of A. (Don't sweat about it too much, but a matrix which is not invertible is sometimes called a **singular matrix**, and an invertible matrix is sometimes referred to as a **non-singular matrix**.)

Invertible matrices

Definition

An $n \times n$ matrix A is **invertible** if there exists an $n \times n$ matrix C such that

$$AC = I_n = CA.$$

If such a C exists, we denote it by A^{-1} and refer to it as the **inverse** of A. (Don't sweat about it too much, but a matrix which is not invertible is sometimes called a **singular matrix**, and an invertible matrix is sometimes referred to as a **non-singular matrix**.)

Example

Let $A = \begin{bmatrix} 7 & 2 \\ 3 & 1 \end{bmatrix}$ and $C = \begin{bmatrix} 1 & -2 \\ -3 & 7 \end{bmatrix}$.

Invertible matrices

Definition

An $n \times n$ matrix A is **invertible** if there exists an $n \times n$ matrix C such that

$$AC = I_n = CA.$$

If such a C exists, we denote it by A^{-1} and refer to it as the **inverse** of A. (Don't sweat about it too much, but a matrix which is not invertible is sometimes called a **singular matrix**, and an invertible matrix is sometimes referred to as a **non-singular matrix**.)

Example

Let $A = \begin{bmatrix} 7 & 2 \\ 3 & 1 \end{bmatrix}$ and $C = \begin{bmatrix} 1 & -2 \\ -3 & 7 \end{bmatrix}$. Then you can check that

$$AC = CA = I_2$$

Thus A is invertible and $A^{-1} = C$.

Examples

Example

If $A = I_2 = \begin{bmatrix} 1 & 0 \\ 0 & 1 \end{bmatrix}$, then A is invertible and $A = A^{-1}$.

Examples

Example

If $A = I_2 = \begin{bmatrix} 1 & 0 \\ 0 & 1 \end{bmatrix}$, then A is invertible and $A = A^{-1}$.

Example

The matrix $A = \begin{bmatrix} 1 & 0 \\ 0 & 0 \end{bmatrix}$ is *not invertible*.

Examples

Example

If $A = I_2 = \begin{bmatrix} 1 & 0 \\ 0 & 1 \end{bmatrix}$, then A is invertible and $A = A^{-1}$.

Example

The matrix $A = \begin{bmatrix} 1 & 0 \\ 0 & 0 \end{bmatrix}$ is *not invertible*. Note that

$$\begin{bmatrix} 1 & 0 \\ 0 & 0 \end{bmatrix} \begin{bmatrix} a & b \\ c & d \end{bmatrix} = \begin{bmatrix} a & b \\ 0 & 0 \end{bmatrix}.$$

Examples

Example

If $A = I_2 = \begin{bmatrix} 1 & 0 \\ 0 & 1 \end{bmatrix}$, then A is invertible and $A = A^{-1}$.

Example

The matrix $A = \begin{bmatrix} 1 & 0 \\ 0 & 0 \end{bmatrix}$ is *not invertible*. Note that

$$\begin{bmatrix} 1 & 0 \\ 0 & 0 \end{bmatrix} \begin{bmatrix} a & b \\ c & d \end{bmatrix} = \begin{bmatrix} a & b \\ 0 & 0 \end{bmatrix}.$$

So no matter what you multiply A by, you can't get the identity matrix.

Inverses for 2×2 matrices

Probably you noticed a relationship between the entries of the matrices in the previous example.

$$A = \begin{bmatrix} 7 & 2 \\ 3 & 1 \end{bmatrix} \text{ and } A^{-1} = \begin{bmatrix} 1 & -2 \\ -3 & 7 \end{bmatrix}.$$

Inverses for 2×2 matrices

Probably you noticed a relationship between the entries of the matrices in the previous example.

$$A = \begin{bmatrix} 7 & 2 \\ 3 & 1 \end{bmatrix} \text{ and } A^{-1} = \begin{bmatrix} 1 & -2 \\ -3 & 7 \end{bmatrix}.$$

There is a rule for 2×2 matrices describing this precisely.

Theorem

Let $A = \begin{bmatrix} a & b \\ c & d \end{bmatrix}$.

Inverses for 2 × 2 matrices

Probably you noticed a relationship between the entries of the matrices in the previous example.

$$A = \begin{bmatrix} 7 & 2 \\ 3 & 1 \end{bmatrix} \text{ and } A^{-1} = \begin{bmatrix} 1 & -2 \\ -3 & 7 \end{bmatrix}.$$

There is a rule for 2 × 2 matrices describing this precisely.

Theorem

Let $A = \begin{bmatrix} a & b \\ c & d \end{bmatrix}$. *If* $ad - bc \neq 0$, *then* A *is invertible and*

$$A^{-1} = \frac{1}{ad - bc} \begin{bmatrix} d & -b \\ -c & a \end{bmatrix}.$$

Inverses for 2×2 matrices

Probably you noticed a relationship between the entries of the matrices in the previous example.

$$A = \begin{bmatrix} 7 & 2 \\ 3 & 1 \end{bmatrix} \text{ and } A^{-1} = \begin{bmatrix} 1 & -2 \\ -3 & 7 \end{bmatrix}.$$

There is a rule for 2×2 matrices describing this precisely.

Theorem

Let $A = \begin{bmatrix} a & b \\ c & d \end{bmatrix}$. If $ad - bc \neq 0$, then A is invertible and

$$A^{-1} = \frac{1}{ad - bc} \begin{bmatrix} d & -b \\ -c & a \end{bmatrix}.$$

If $ad - bc = 0$ then A is not invertible.

Inverses for 2 × 2 matrices

Probably you noticed a relationship between the entries of the matrices in the previous example.

$$A = \begin{bmatrix} 7 & 2 \\ 3 & 1 \end{bmatrix} \text{ and } A^{-1} = \begin{bmatrix} 1 & -2 \\ -3 & 7 \end{bmatrix}.$$

There is a rule for 2×2 matrices describing this precisely.

Theorem

Let $A = \begin{bmatrix} a & b \\ c & d \end{bmatrix}$*. If* $ad - bc \neq 0$*, then* A *is invertible and*

$$A^{-1} = \frac{1}{ad - bc} \begin{bmatrix} d & -b \\ -c & a \end{bmatrix}.$$

If $ad - bc = 0$ *then* A *is not invertible.*

The quantity $ad - bc$ (diagonal product minus off-diagonal product) is called the **determinant** of A.

Inverses for 2×2 matrices, ctd.

$$A^{-1} = \frac{1}{ad - bc} \begin{bmatrix} d & -b \\ -c & a \end{bmatrix}$$

Inverses for 2×2 matrices, ctd.

$$A^{-1} = \frac{1}{ad - bc} \begin{bmatrix} d & -b \\ -c & a \end{bmatrix}$$

Example

Let $B = \begin{bmatrix} 1 & 1 \\ 2 & 2 \end{bmatrix}$.

Inverses for 2×2 matrices, ctd.

$$A^{-1} = \frac{1}{ad - bc} \begin{bmatrix} d & -b \\ -c & a \end{bmatrix}$$

Example

Let $B = \begin{bmatrix} 1 & 1 \\ 2 & 2 \end{bmatrix}$. Is B invertible?

Inverses for 2×2 matrices, ctd.

$$A^{-1} = \frac{1}{ad - bc} \begin{bmatrix} d & -b \\ -c & a \end{bmatrix}$$

Example

Let $B = \begin{bmatrix} 1 & 1 \\ 2 & 2 \end{bmatrix}$. Is B invertible? We compute the determinant: $(1)(2) - (2)(1) = 0$.

Inverses for 2×2 matrices, ctd.

$$A^{-1} = \frac{1}{ad - bc} \begin{bmatrix} d & -b \\ -c & a \end{bmatrix}$$

Example

Let $B = \begin{bmatrix} 1 & 1 \\ 2 & 2 \end{bmatrix}$. Is B invertible? We compute the determinant: $(1)(2) - (2)(1) = 0$. The determinant is 0, so B is not invertible.

Inverses for 2×2 matrices, ctd.

$$A^{-1} = \frac{1}{ad - bc} \begin{bmatrix} d & -b \\ -c & a \end{bmatrix}$$

Example

Let $B = \begin{bmatrix} 1 & 1 \\ 2 & 2 \end{bmatrix}$. Is B invertible? We compute the determinant: $(1)(2) - (2)(1) = 0$. The determinant is 0, so B is not invertible.

Example

Let $A = \begin{bmatrix} 5 & 2 \\ 2 & 1 \end{bmatrix}$.

Inverses for 2×2 matrices, ctd.

$$A^{-1} = \frac{1}{ad - bc} \begin{bmatrix} d & -b \\ -c & a \end{bmatrix}$$

Example

Let $B = \begin{bmatrix} 1 & 1 \\ 2 & 2 \end{bmatrix}$. Is B invertible? We compute the determinant: $(1)(2) - (2)(1) = 0$. The determinant is 0, so B is not invertible.

Example

Let $A = \begin{bmatrix} 5 & 2 \\ 2 & 1 \end{bmatrix}$. Compute the determinant: $(5)(1) - (2)(2) = 1$.

So A is invertible.

Inverses for 2×2 matrices, ctd.

$$A^{-1} = \frac{1}{ad - bc} \begin{bmatrix} d & -b \\ -c & a \end{bmatrix}$$

Example

Let $B = \begin{bmatrix} 1 & 1 \\ 2 & 2 \end{bmatrix}$. Is B invertible? We compute the determinant: $(1)(2) - (2)(1) = 0$. The determinant is 0, so B is not invertible.

Example

Let $A = \begin{bmatrix} 5 & 2 \\ 2 & 1 \end{bmatrix}$. Compute the determinant: $(5)(1) - (2)(2) = 1$.
So A is invertible. We have $A^{-1} = \frac{1}{1} \begin{bmatrix} 1 & -2 \\ -2 & 5 \end{bmatrix} = \begin{bmatrix} 1 & -2 \\ -2 & 5 \end{bmatrix}$.

Using inverse matrices to solve equations

You can solve equations with the matrix inverse.

Using inverse matrices to solve equations

You can solve equations with the matrix inverse.

Theorem

Let A be an invertible $n \times n$ matrix. Then for each $\mathbf{b} \in \mathbb{R}^n$, the equation $A\mathbf{x} = \mathbf{b}$ is consistent and has unique solution $\mathbf{x} = A^{-1}\mathbf{b}$.

Using inverse matrices to solve equations

You can solve equations with the matrix inverse.

Theorem

Let A be an invertible $n \times n$ matrix. Then for each $\mathbf{b} \in \mathbb{R}^n$, the equation $A\mathbf{x} = \mathbf{b}$ is consistent and has unique solution $\mathbf{x} = A^{-1}\mathbf{b}$.

Proof.

First we show that $A^{-1}\mathbf{b}$ is a solution.

Using inverse matrices to solve equations

You can solve equations with the matrix inverse.

Theorem

Let A be an invertible $n \times n$ matrix. Then for each $\mathbf{b} \in \mathbb{R}^n$, the equation $A\mathbf{x} = \mathbf{b}$ is consistent and has unique solution $\mathbf{x} = A^{-1}\mathbf{b}$.

Proof.

First we show that $A^{-1}\mathbf{b}$ is a solution.Check:
$A(A^{-1}\mathbf{b}) = (AA^{-1})\mathbf{b}$, and $AA^{-1} = I_n$. So $A(A^{-1}\mathbf{b}) = I_n\mathbf{b} = \mathbf{b}$.
Thus $\mathbf{x} = A^{-1}\mathbf{b}$ is a solution to $A\mathbf{x} = \mathbf{b}$.

Using inverse matrices to solve equations

You can solve equations with the matrix inverse.

Theorem

Let A be an invertible $n \times n$ matrix. Then for each $\mathbf{b} \in \mathbb{R}^n$, the equation $A\mathbf{x} = \mathbf{b}$ is consistent and has unique solution $\mathbf{x} = A^{-1}\mathbf{b}$.

Proof.

First we show that $A^{-1}\mathbf{b}$ is a solution.Check:
$A(A^{-1}\mathbf{b}) = (AA^{-1})\mathbf{b}$, and $AA^{-1} = I_n$. So $A(A^{-1}\mathbf{b}) = I_n\mathbf{b} = \mathbf{b}$.
Thus $\mathbf{x} = A^{-1}\mathbf{b}$ is a solution to $A\mathbf{x} = \mathbf{b}$.
If $\mathbf{x}$ is a solution to $A\mathbf{x} = \mathbf{b}$, then multiply both sides of the equation to obtain $A^{-1}(A\mathbf{x}) = A^{-1}\mathbf{b}$.

You can solve equations with the matrix inverse.

Theorem

Let A be an invertible $n \times n$ matrix. Then for each $\mathbf{b} \in \mathbb{R}^n$, the equation $A\mathbf{x} = \mathbf{b}$ is consistent and has unique solution $\mathbf{x} = A^{-1}\mathbf{b}$.

Proof.

First we show that $A^{-1}\mathbf{b}$ is a solution.Check:
$A(A^{-1}\mathbf{b}) = (AA^{-1})\mathbf{b}$, and $AA^{-1} = I_n$. So $A(A^{-1}\mathbf{b}) = I_n\mathbf{b} = \mathbf{b}$.
Thus $\mathbf{x} = A^{-1}\mathbf{b}$ is a solution to $A\mathbf{x} = \mathbf{b}$.
If $\mathbf{x}$ is a solution to $A\mathbf{x} = \mathbf{b}$, then multiply both sides of the equation to obtain $A^{-1}(A\mathbf{x}) = A^{-1}\mathbf{b}$. Then rearrange and cancel on the left to obtain $\mathbf{x} = A^{-1}\mathbf{b}$ □

Example: solving systems with inverses

Lets solve

$$\begin{aligned} 3x + y &= 11 \\ 3x + 2y &= 5 \end{aligned}$$

using inverses of matrices.

Example: solving systems with inverses

Lets solve

$$\begin{aligned} 3x + y &= 11 \\ 3x + 2y &= 5 \end{aligned}$$

using inverses of matrices. This is equivalent to solving $A\mathbf{x} = \mathbf{b}$, where $A = \begin{bmatrix} 3 & 1 \\ 3 & 2 \end{bmatrix}$ and $\mathbf{b} = \begin{bmatrix} 11 \\ 5 \end{bmatrix}$.

Example: solving systems with inverses

Lets solve

$$\begin{aligned} 3x + y &= 11 \\ 3x + 2y &= 5 \end{aligned}$$

using inverses of matrices. This is equivalent to solving $A\mathbf{x} = \mathbf{b}$, where $A = \begin{bmatrix} 3 & 1 \\ 3 & 2 \end{bmatrix}$ and $\mathbf{b} = \begin{bmatrix} 11 \\ 5 \end{bmatrix}$. The determinant of A is

$(3)(2) - (3) = 3.$

Example: solving systems with inverses

Lets solve

$$\begin{aligned} 3x + y &= 11 \\ 3x + 2y &= 5 \end{aligned}$$

using inverses of matrices. This is equivalent to solving $A\mathbf{x} = \mathbf{b}$, where $A = \begin{bmatrix} 3 & 1 \\ 3 & 2 \end{bmatrix}$ and $\mathbf{b} = \begin{bmatrix} 11 \\ 5 \end{bmatrix}$. The determinant of A is $(3)(2) - (3) = 3$. Thus $A^{-1} = \frac{1}{3}\begin{bmatrix} 2 & -1 \\ -3 & 3 \end{bmatrix} = \begin{bmatrix} 2/3 & -1/3 \\ -1 & 1 \end{bmatrix}$.

Example: solving systems with inverses

Lets solve

$$3x + \quad y = 11$$
$$3x + 2y = 5$$

using inverses of matrices. This is equivalent to solving $A\mathbf{x} = \mathbf{b}$, where $A = \begin{bmatrix} 3 & 1 \\ 3 & 2 \end{bmatrix}$ and $\mathbf{b} = \begin{bmatrix} 11 \\ 5 \end{bmatrix}$. The determinant of A is $(3)(2) - (3) = 3$. Thus $A^{-1} = \frac{1}{3}\begin{bmatrix} 2 & -1 \\ -3 & 3 \end{bmatrix} = \begin{bmatrix} 2/3 & -1/3 \\ -1 & 1 \end{bmatrix}$.
Thus the solution $\mathbf{x}$ to the above equation is

$$\mathbf{x} = A^{-1}\mathbf{b}$$

Example: solving systems with inverses

Lets solve

$$3x + y = 11$$
$$3x + 2y = 5$$

using inverses of matrices. This is equivalent to solving $A\mathbf{x} = \mathbf{b}$, where $A = \begin{bmatrix} 3 & 1 \\ 3 & 2 \end{bmatrix}$ and $\mathbf{b} = \begin{bmatrix} 11 \\ 5 \end{bmatrix}$. The determinant of A is $(3)(2) - (3) = 3$. Thus $A^{-1} = \frac{1}{3}\begin{bmatrix} 2 & -1 \\ -3 & 3 \end{bmatrix} = \begin{bmatrix} 2/3 & -1/3 \\ -1 & 1 \end{bmatrix}$.
Thus the solution $\mathbf{x}$ to the above equation is

$$\mathbf{x} = A^{-1}\mathbf{b} = \begin{bmatrix} 2/3 & -1/3 \\ -1 & 1 \end{bmatrix} \begin{bmatrix} 11 \\ 5 \end{bmatrix}$$

Example: solving systems with inverses

Lets solve

$$3x + \quad y = 11$$
$$3x + 2y = 5$$

using inverses of matrices. This is equivalent to solving $A\mathbf{x} = \mathbf{b}$, where $A = \begin{bmatrix} 3 & 1 \\ 3 & 2 \end{bmatrix}$ and $\mathbf{b} = \begin{bmatrix} 11 \\ 5 \end{bmatrix}$. The determinant of A is $(3)(2) - (3) = 3$. Thus $A^{-1} = \frac{1}{3}\begin{bmatrix} 2 & -1 \\ -3 & 3 \end{bmatrix} = \begin{bmatrix} 2/3 & -1/3 \\ -1 & 1 \end{bmatrix}$.
Thus the solution $\mathbf{x}$ to the above equation is

$$\mathbf{x} = A^{-1}\mathbf{b} = \begin{bmatrix} 2/3 & -1/3 \\ -1 & 1 \end{bmatrix} \begin{bmatrix} 11 \\ 5 \end{bmatrix} = \begin{bmatrix} 22/3 - 5/3 \\ -11 + 5 \end{bmatrix}$$

Example: solving systems with inverses

Lets solve

$$\begin{aligned} 3x + y &= 11 \\ 3x + 2y &= 5 \end{aligned}$$

using inverses of matrices. This is equivalent to solving $A\mathbf{x} = \mathbf{b}$, where $A = \begin{bmatrix} 3 & 1 \\ 3 & 2 \end{bmatrix}$ and $\mathbf{b} = \begin{bmatrix} 11 \\ 5 \end{bmatrix}$. The determinant of A is $(3)(2) - (3) = 3$. Thus $A^{-1} = \frac{1}{3}\begin{bmatrix} 2 & -1 \\ -3 & 3 \end{bmatrix} = \begin{bmatrix} 2/3 & -1/3 \\ -1 & 1 \end{bmatrix}$.

Thus the solution $\mathbf{x}$ to the above equation is

$$\mathbf{x} = A^{-1}\mathbf{b} = \begin{bmatrix} 2/3 & -1/3 \\ -1 & 1 \end{bmatrix}\begin{bmatrix} 11 \\ 5 \end{bmatrix} = \begin{bmatrix} 22/3 - 5/3 \\ -11 + 5 \end{bmatrix} = \begin{bmatrix} 17/3 \\ -6 \end{bmatrix}.$$

Example: solving systems with inverses

Lets solve

$$3x + y = 11$$
$$3x + 2y = 5$$

using inverses of matrices. This is equivalent to solving $A\mathbf{x} = \mathbf{b}$, where $A = \begin{bmatrix} 3 & 1 \\ 3 & 2 \end{bmatrix}$ and $\mathbf{b} = \begin{bmatrix} 11 \\ 5 \end{bmatrix}$. The determinant of A is $(3)(2) - (3) = 3$. Thus $A^{-1} = \frac{1}{3}\begin{bmatrix} 2 & -1 \\ -3 & 3 \end{bmatrix} = \begin{bmatrix} 2/3 & -1/3 \\ -1 & 1 \end{bmatrix}$.
Thus the solution $\mathbf{x}$ to the above equation is

$$\mathbf{x} = A^{-1}\mathbf{b} = \begin{bmatrix} 2/3 & -1/3 \\ -1 & 1 \end{bmatrix}\begin{bmatrix} 11 \\ 5 \end{bmatrix} = \begin{bmatrix} 22/3 - 5/3 \\ -11 + 5 \end{bmatrix} = \begin{bmatrix} 17/3 \\ -6 \end{bmatrix}.$$

Thus the unique solution is $\mathbf{x} = \begin{bmatrix} 17/3 \\ -6 \end{bmatrix}$.

Example: solving systems with inverses

Lets solve

$$\begin{aligned} 3x + \quad y &= 11 \\ 3x + 2y &= 5 \end{aligned}$$

using inverses of matrices. This is equivalent to solving $A\mathbf{x} = \mathbf{b}$, where $A = \begin{bmatrix} 3 & 1 \\ 3 & 2 \end{bmatrix}$ and $\mathbf{b} = \begin{bmatrix} 11 \\ 5 \end{bmatrix}$. The determinant of A is $(3)(2) - (3) = 3$. Thus $A^{-1} = \frac{1}{3}\begin{bmatrix} 2 & -1 \\ -3 & 3 \end{bmatrix} = \begin{bmatrix} 2/3 & -1/3 \\ -1 & 1 \end{bmatrix}$.
Thus the solution $\mathbf{x}$ to the above equation is

$$\mathbf{x} = A^{-1}\mathbf{b} = \begin{bmatrix} 2/3 & -1/3 \\ -1 & 1 \end{bmatrix}\begin{bmatrix} 11 \\ 5 \end{bmatrix} = \begin{bmatrix} 22/3 - 5/3 \\ -11 + 5 \end{bmatrix} = \begin{bmatrix} 17/3 \\ -6 \end{bmatrix}.$$

Thus the unique solution is $\mathbf{x} = \begin{bmatrix} 17/3 \\ -6 \end{bmatrix}$.So $x = 17/3$ and $y = -6$.

Properties of invertible matrices

Properties of invertible matrices

Theorem

1. *If A is an invertible matrix, then A^{-1} is also invertible, and the inverse of A^{-1} is A.*

Properties of invertible matrices

Theorem

1. *If A is an invertible matrix, then A^{-1} is also invertible, and the inverse of A^{-1} is A.*
2. *If A and B are $n \times n$ invertible matrices, then so is AB, and the inverse of AB is $B^{-1}A^{-1}$ [Similar to the formula for the transpose of a product]*

Properties of invertible matrices

Theorem

1. *If A is an invertible matrix, then A^{-1} is also invertible, and the inverse of A^{-1} is A.*
2. *If A and B are $n \times n$ invertible matrices, then so is AB, and the inverse of AB is $B^{-1}A^{-1}$ [Similar to the formula for the transpose of a product]*
3. *The transpose of an invertible matrix is invertible, and $(A^T)^{-1} = (A^{-1})^T$.*

Properties of invertible matrices

Theorem

1. *If A is an invertible matrix, then A^{-1} is also invertible, and the inverse of A^{-1} is A.*
2. *If A and B are $n \times n$ invertible matrices, then so is AB, and the inverse of AB is $B^{-1}A^{-1}$ [Similar to the formula for the transpose of a product]*
3. *The transpose of an invertible matrix is invertible, and $(A^T)^{-1} = (A^{-1})^T$.*

Proof.

Proof in the book is uncomplicated but you should read it. You really only need associativity of matrix multiplication, the fact that $I_nA = A = AI_n$ for $n \times n$ matrices, and properties of the transpose. □

Properties of invertible matrices

Theorem

1. *If A is an invertible matrix, then A^{-1} is also invertible, and the inverse of A^{-1} is A.*
2. *If A and B are $n \times n$ invertible matrices, then so is AB, and the inverse of AB is $B^{-1}A^{-1}$ [Similar to the formula for the transpose of a product]*
3. *The transpose of an invertible matrix is invertible, and $(A^T)^{-1} = (A^{-1})^T$.*

Proof.

Proof in the book is uncomplicated but you should read it. You really only need associativity of matrix multiplication, the fact that $I_nA = A = AI_n$ for $n \times n$ matrices, and properties of the transpose. □

In general, if $A_1, A_2, \ldots, A_k$ are a bunch of invertible $n \times n$ matrices then so is $A_1A_2 \cdots A_k$ and $(A_1A_2 \cdots A_k)^{-1} = A_k^{-1} \cdots A_2^{-1}A_1^{-1}$

Elementary matrices and row operations

Recall the elementary row operations: interchange, replacement, scaling.

Elementary matrices and row operations

Recall the elementary row operations: interchange, replacement, scaling. We want to reformulate them so that performing an elementary row operation on the square matrix A is the same as taking the product EA by some invertible matrix E. This will allow us effective computation of matrix inverses.

Elementary matrices and row operations

Recall the elementary row operations: interchange, replacement, scaling. We want to reformulate them so that performing an elementary row operation on the square matrix A is the same as taking the product EA by some invertible matrix E. This will allow us effective computation of matrix inverses.

Definition

An **elementary matrix** is a matrix obtained by performing an elementary row operation on an identity matrix.

Elementary matrices and row operations

Recall the elementary row operations: interchange, replacement, scaling. We want to reformulate them so that performing an elementary row operation on the square matrix A is the same as taking the product EA by some invertible matrix E. This will allow us effective computation of matrix inverses.

Definition

An **elementary matrix** is a matrix obtained by performing an elementary row operation on an identity matrix.

Example

$$\begin{bmatrix} 0 & 1 \\ 1 & 0 \end{bmatrix}$$

Elementary matrices and row operations

Recall the elementary row operations: interchange, replacement, scaling. We want to reformulate them so that performing an elementary row operation on the square matrix A is the same as taking the product EA by some invertible matrix E. This will allow us effective computation of matrix inverses.

Definition

An **elementary matrix** is a matrix obtained by performing an elementary row operation on an identity matrix.

Example

$$\begin{bmatrix} 0 & 1 \\ 1 & 0 \end{bmatrix} \qquad \begin{bmatrix} 1 & 0 & 0 \\ 2 & 1 & 0 \\ 0 & 0 & 1 \end{bmatrix}$$

Elementary matrices and row operations

Recall the elementary row operations: interchange, replacement, scaling. We want to reformulate them so that performing an elementary row operation on the square matrix A is the same as taking the product EA by some invertible matrix E. This will allow us effective computation of matrix inverses.

Definition

An **elementary matrix** is a matrix obtained by performing an elementary row operation on an identity matrix.

Example

$$\begin{bmatrix} 0 & 1 \\ 1 & 0 \end{bmatrix} \qquad \begin{bmatrix} 1 & 0 & 0 \\ 2 & 1 & 0 \\ 0 & 0 & 1 \end{bmatrix} \qquad \begin{bmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 5 \end{bmatrix}$$

Elementary matrices and row operations

Recall the elementary row operations: interchange, replacement, scaling. We want to reformulate them so that performing an elementary row operation on the square matrix A is the same as taking the product EA by some invertible matrix E. This will allow us effective computation of matrix inverses.

Definition

An **elementary matrix** is a matrix obtained by performing an elementary row operation on an identity matrix.

Example

$$\begin{bmatrix} 0 & 1 \\ 1 & 0 \end{bmatrix} \qquad \begin{bmatrix} 1 & 0 & 0 \\ 2 & 1 & 0 \\ 0 & 0 & 1 \end{bmatrix} \qquad \begin{bmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 5 \end{bmatrix}$$

are all elementary matrices.

Elementary matrices are invertible

Remember that the elementary row operations can be reversed:

Elementary matrices are invertible

Remember that the elementary row operations can be reversed:

1. interchanging rows i_1 and i_2 is reversed by interchanging the rows again.

Elementary matrices are invertible

Remember that the elementary row operations can be reversed:

1. interchanging rows i_1 and i_2 is reversed by interchanging the rows again.
2. scaling a row by c is reversed by scaling the same row by $\frac{1}{c}$

Elementary matrices are invertible

Remember that the elementary row operations can be reversed:

1. interchanging rows i_1 and i_2 is reversed by interchanging the rows again.
2. scaling a row by c is reversed by scaling the same row by $\frac{1}{c}$
3. replacing row i_1 with c times row i_2 is reversed by subtracting c times row i_2 from row i_1

Elementary matrices are invertible

Remember that the elementary row operations can be reversed:

1. interchanging rows i_1 and i_2 is reversed by interchanging the rows again.
2. scaling a row by c is reversed by scaling the same row by $\frac{1}{c}$
3. replacing row i_1 with c times row i_2 is reversed by subtracting c times row i_2 from row i_1

This gives us that every elementary matrix E corresponding to a row operation is invertible, and E^{-1} is the elementary matrix corresponding to the reverse elementary row operation.

Elementry matrices are invertible

Example

Let $E = \begin{bmatrix} 1 & 0 \\ 5 & 1 \end{bmatrix}$. Then E is the elementary matrix corresponding to the row operation "add 5 times the first row to the second row."

Example

Let $E = \begin{bmatrix} 1 & 0 \\ 5 & 1 \end{bmatrix}$. Then E is the elementary matrix corresponding to the row operation "add 5 times the first row to the second row." Then its inverse is the elementary matrix corresponding to the row operation "subtract 5 times the first row from the second row."

Example

Let $E = \begin{bmatrix} 1 & 0 \\ 5 & 1 \end{bmatrix}$. Then E is the elementary matrix corresponding to the row operation "add 5 times the first row to the second row." Then its inverse is the elementary matrix corresponding to the row operation "subtract 5 times the first row from the second row." That is,

$$E^{-1} = \begin{bmatrix} 1 & 0 \\ -5 & 1 \end{bmatrix}.$$

Elementary matrices in action

Example

Let E_1 be the elementary matrix obtained by adding the first row of I_2 to its second row.

$$E_1 = \begin{bmatrix} 1 & 0 \\ 1 & 1 \end{bmatrix}.$$

Elementary matrices in action

Example

Let E_1 be the elementary matrix obtained by adding the first row of I_2 to its second row.

$$E_1 = \begin{bmatrix} 1 & 0 \\ 1 & 1 \end{bmatrix}.$$

Then for any 2×2 matrix $A = \begin{bmatrix} a & b \\ c & d \end{bmatrix}$ we have

Elementary matrices in action

Example

Let E_1 be the elementary matrix obtained by adding the first row of I_2 to its second row.

$$E_1 = \begin{bmatrix} 1 & 0 \\ 1 & 1 \end{bmatrix}.$$

Then for any 2×2 matrix $A = \begin{bmatrix} a & b \\ c & d \end{bmatrix}$ we have

$$E_1 A$$

Example

Let E_1 be the elementary matrix obtained by adding the first row of I_2 to its second row.

$$E_1 = \begin{bmatrix} 1 & 0 \\ 1 & 1 \end{bmatrix}.$$

Then for any 2×2 matrix $A = \begin{bmatrix} a & b \\ c & d \end{bmatrix}$ we have

$$E_1 A = \begin{bmatrix} 1 & 0 \\ 1 & 1 \end{bmatrix} \begin{bmatrix} a & b \\ c & d \end{bmatrix}$$

Elementary matrices in action

Example

Let E_1 be the elementary matrix obtained by adding the first row of I_2 to its second row.

$$E_1 = \begin{bmatrix} 1 & 0 \\ 1 & 1 \end{bmatrix}.$$

Then for any 2×2 matrix $A = \begin{bmatrix} a & b \\ c & d \end{bmatrix}$ we have

$$E_1 A = \begin{bmatrix} 1 & 0 \\ 1 & 1 \end{bmatrix} \begin{bmatrix} a & b \\ c & d \end{bmatrix} = \begin{bmatrix} a & b \\ a+c & b+d \end{bmatrix}.$$

Invertibility and elementary row ops.

Theorem

Let A be an $n \times n$ matrix. Then A is invertible if and only if A is row equivalent to I_n, and any sequence of row operations that transforms A into I applied in the same order to I transforms I into A^{-1}.

Theorem

Let A be an $n \times n$ matrix. Then A is invertible if and only if A is row equivalent to I_n, and any sequence of row operations that transforms A into I applied in the same order to I transforms I into A^{-1}. Put another way, if $E_1, E_2, \ldots, E_k$ are the elementary matrices corresponding to the row operations transforming A into I, so that

$$E_1 E_2 \ldots E_k A = I,$$

Invertibility and elementary row ops.

Theorem

Let A be an $n \times n$ matrix. Then A is invertible if and only if A is row equivalent to I_n, and any sequence of row operations that transforms A into I applied in the same order to I transforms I into A^{-1}. Put another way, if $E_1, E_2, \ldots, E_k$ are the elementary matrices corresponding to the row operations transforming A into I, so that

$$E_1 E_2 \ldots E_k A = I,$$

then we necessarily have that

$$E_1 E_2 \ldots E_k I = E_1 \ldots E_k = A^{-1}.$$

Matrix inversion algorithm

Matrix inversion algorithm

The theorem in the preceding slide tells us how to compute the inverse of *any invertible matrix*, and it encodes a way to check if an $n \times n$ matrix is invertible, all through row reduction.

Matrix inversion algorithm

The theorem in the preceding slide tells us how to compute the inverse of *any invertible matrix*, and it encodes a way to check if an $n \times n$ matrix is invertible, all through row reduction.

Fact

Let A be an $n \times n$ matrix and form the augmented matrix

$$[A \quad I_n].$$

Row reduce this matrix as usual. If the reduced echelon form looks like

$$[I_n \quad C]$$

then A is invertible and $C = A^{-1}$.

Matrix inversion algorithm

The theorem in the preceding slide tells us how to compute the inverse of *any invertible matrix*, and it encodes a way to check if an $n \times n$ matrix is invertible, all through row reduction.

Fact

Let A be an $n \times n$ matrix and form the augmented matrix

$$[A \quad I_n].$$

Row reduce this matrix as usual. If the reduced echelon form looks like

$$[I_n \quad C]$$

then A is invertible and $C = A^{-1}$. If the reduced echelon form looks any other way, then A is not invertible (in fact, you can decide if A is invertible just by transforming to echelon form and seeing if there are any rows that have n 0s in the left half.

Example of MIA

Let's find determine if the matrix $A = \begin{bmatrix} 1 & 2 & 3 \\ 3 & 1 & 2 \\ 2 & 3 & 1 \end{bmatrix}$ is invertible and find its inverse, if possible. Form the augmented system

$$\begin{bmatrix} 1 & 2 & 3 & 1 & 0 & 0 \\ 3 & 1 & 2 & 0 & 1 & 0 \\ 2 & 3 & 1 & 0 & 0 & 1 \end{bmatrix}.$$

Ex. of MIA, pt. II

$$\begin{bmatrix} 1 & 2 & 3 & 1 & 0 & 0 \\ 3 & 1 & 2 & 0 & 1 & 0 \\ 2 & 3 & 1 & 0 & 0 & 1 \end{bmatrix}$$

Ex. of MIA, pt. II

$$\begin{bmatrix} 1 & 2 & 3 & 1 & 0 & 0 \\ 3 & 1 & 2 & 0 & 1 & 0 \\ 2 & 3 & 1 & 0 & 0 & 1 \end{bmatrix} \to \begin{bmatrix} 1 & 2 & 3 & 1 & 0 & 0 \\ 0 & -5 & -7 & -3 & 1 & 0 \\ 0 & -1 & -5 & -2 & 0 & 1 \end{bmatrix}$$

Ex. of MIA, pt. II

$$\begin{bmatrix} 1 & 2 & 3 & 1 & 0 & 0 \\ 3 & 1 & 2 & 0 & 1 & 0 \\ 2 & 3 & 1 & 0 & 0 & 1 \end{bmatrix} \to \begin{bmatrix} 1 & 2 & 3 & 1 & 0 & 0 \\ 0 & -5 & -7 & -3 & 1 & 0 \\ 0 & -1 & -5 & -2 & 0 & 1 \end{bmatrix}$$

$$\begin{bmatrix} 1 & 2 & 3 & 1 & 0 & 0 \\ 0 & -5 & -7 & -3 & 1 & 0 \\ 0 & -1 & -5 & -2 & 0 & 1 \end{bmatrix}$$

Ex. of MIA, pt. II

$$\begin{bmatrix} 1 & 2 & 3 & 1 & 0 & 0 \\ 3 & 1 & 2 & 0 & 1 & 0 \\ 2 & 3 & 1 & 0 & 0 & 1 \end{bmatrix} \to \begin{bmatrix} 1 & 2 & 3 & 1 & 0 & 0 \\ 0 & -5 & -7 & -3 & 1 & 0 \\ 0 & -1 & -5 & -2 & 0 & 1 \end{bmatrix}$$

$$\begin{bmatrix} 1 & 2 & 3 & 1 & 0 & 0 \\ 0 & -5 & -7 & -3 & 1 & 0 \\ 0 & -1 & -5 & -2 & 0 & 1 \end{bmatrix} \to \begin{bmatrix} 1 & 2 & 3 & 1 & 0 & 0 \\ 0 & 1 & 5 & 2 & 0 & -1 \\ 0 & -5 & -7 & -3 & 1 & 0 \end{bmatrix}$$

Ex. of MIA, pt. II

$$\begin{bmatrix} 1 & 2 & 3 & 1 & 0 & 0 \\ 3 & 1 & 2 & 0 & 1 & 0 \\ 2 & 3 & 1 & 0 & 0 & 1 \end{bmatrix} \to \begin{bmatrix} 1 & 2 & 3 & 1 & 0 & 0 \\ 0 & -5 & -7 & -3 & 1 & 0 \\ 0 & -1 & -5 & -2 & 0 & 1 \end{bmatrix}$$

$$\begin{bmatrix} 1 & 2 & 3 & 1 & 0 & 0 \\ 0 & -5 & -7 & -3 & 1 & 0 \\ 0 & -1 & -5 & -2 & 0 & 1 \end{bmatrix} \to \begin{bmatrix} 1 & 2 & 3 & 1 & 0 & 0 \\ 0 & 1 & 5 & 2 & 0 & -1 \\ 0 & -5 & -7 & -3 & 1 & 0 \end{bmatrix}$$

$$\begin{bmatrix} 1 & 2 & 3 & 1 & 0 & 0 \\ 0 & 1 & 5 & 2 & 0 & -1 \\ 0 & -5 & -7 & -3 & 1 & 0 \end{bmatrix}$$

Ex. of MIA, pt. II

$$\begin{bmatrix} 1 & 2 & 3 & 1 & 0 & 0 \\ 3 & 1 & 2 & 0 & 1 & 0 \\ 2 & 3 & 1 & 0 & 0 & 1 \end{bmatrix} \to \begin{bmatrix} 1 & 2 & 3 & 1 & 0 & 0 \\ 0 & -5 & -7 & -3 & 1 & 0 \\ 0 & -1 & -5 & -2 & 0 & 1 \end{bmatrix}$$

$$\begin{bmatrix} 1 & 2 & 3 & 1 & 0 & 0 \\ 0 & -5 & -7 & -3 & 1 & 0 \\ 0 & -1 & -5 & -2 & 0 & 1 \end{bmatrix} \to \begin{bmatrix} 1 & 2 & 3 & 1 & 0 & 0 \\ 0 & 1 & 5 & 2 & 0 & -1 \\ 0 & -5 & -7 & -3 & 1 & 0 \end{bmatrix}$$

$$\begin{bmatrix} 1 & 2 & 3 & 1 & 0 & 0 \\ 0 & 1 & 5 & 2 & 0 & -1 \\ 0 & -5 & -7 & -3 & 1 & 0 \end{bmatrix} \to \begin{bmatrix} 1 & 2 & 3 & 1 & 0 & 0 \\ 0 & 1 & 5 & 2 & 0 & -1 \\ 0 & 0 & 18 & 7 & 1 & -5 \end{bmatrix}.$$

Now we can pause at echelon form to say that there are no bad rows.

Row reduction details

We keep row reducing the entire augmented matrix

$$\begin{bmatrix} 1 & 2 & 3 & 1 & 0 & 0 \\ 0 & 1 & 5 & 2 & 0 & -1 \\ 0 & 0 & 18 & 7 & 1 & -5 \end{bmatrix} \to \begin{bmatrix} 1 & 2 & 3 & 1 & 0 & 0 \\ 0 & 1 & 5 & 2 & 0 & -1 \\ 0 & 0 & 1 & 7/18 & 1/18 & -5/18 \end{bmatrix}$$

$$\begin{bmatrix} 1 & 2 & 3 & 1 & 0 & 0 \\ 0 & 1 & 5 & 2 & 0 & -1 \\ 0 & 0 & 1 & \frac{7}{18} & \frac{1}{18} & \frac{-5}{18} \end{bmatrix}$$

We keep row reducing the entire augmented matrix

$$\begin{bmatrix} 1 & 2 & 3 & 1 & 0 & 0 \\ 0 & 1 & 5 & 2 & 0 & -1 \\ 0 & 0 & 18 & 7 & 1 & -5 \end{bmatrix} \to \begin{bmatrix} 1 & 2 & 3 & 1 & 0 & 0 \\ 0 & 1 & 5 & 2 & 0 & -1 \\ 0 & 0 & 1 & 7/18 & 1/18 & -5/18 \end{bmatrix}$$

$$\begin{bmatrix} 1 & 2 & 3 & 1 & 0 & 0 \\ 0 & 1 & 5 & 2 & 0 & -1 \\ 0 & 0 & 1 & \frac{7}{18} & \frac{1}{18} & \frac{-5}{18} \end{bmatrix} \to \begin{bmatrix} 1 & 2 & 0 & -3/18 & -3/18 & 15/18 \\ 0 & 1 & 0 & 1/18 & -5/18 & 7/18 \\ 0 & 0 & 1 & 7/18 & 1/18 & -5/18 \end{bmatrix}$$

$$\begin{bmatrix} 1 & 2 & 0 & -\frac{3}{18} & \frac{-3}{18} & \frac{15}{18} \\ 0 & 1 & 0 & \frac{1}{18} & \frac{-5}{18} & \frac{-13}{9} \\ 0 & 0 & 1 & \frac{7}{18} & \frac{1}{18} & \frac{-5}{18} \end{bmatrix}$$

Row reduction details

We keep row reducing the entire augmented matrix

$$\begin{bmatrix} 1 & 2 & 3 & 1 & 0 & 0 \\ 0 & 1 & 5 & 2 & 0 & -1 \\ 0 & 0 & 18 & 7 & 1 & -5 \end{bmatrix} \to \begin{bmatrix} 1 & 2 & 3 & 1 & 0 & 0 \\ 0 & 1 & 5 & 2 & 0 & -1 \\ 0 & 0 & 1 & 7/18 & 1/18 & -5/18 \end{bmatrix}$$

$$\begin{bmatrix} 1 & 2 & 3 & 1 & 0 & 0 \\ 0 & 1 & 5 & 2 & 0 & -1 \\ 0 & 0 & 1 & \frac{7}{18} & \frac{1}{18} & \frac{-5}{18} \end{bmatrix} \to \begin{bmatrix} 1 & 2 & 0 & -3/18 & -3/18 & 15/18 \\ 0 & 1 & 0 & 1/18 & -5/18 & 7/18 \\ 0 & 0 & 1 & 7/18 & 1/18 & -5/18 \end{bmatrix}$$

$$\begin{bmatrix} 1 & 2 & 0 & -\frac{3}{18} & \frac{-3}{18} & \frac{15}{18} \\ 0 & 1 & 0 & \frac{1}{18} & \frac{-5}{18} & \frac{-13}{9} \\ 0 & 0 & 1 & \frac{7}{18} & \frac{1}{18} & \frac{-5}{18} \end{bmatrix} \to \begin{bmatrix} 1 & 0 & 0 & -5/18 & 7/8 & 1/18 \\ 0 & 1 & 0 & 1/18 & -5/18 & 7/18 \\ 0 & 0 & 1 & 7/18 & 1/18 & -5/18 \end{bmatrix}$$

This is in reduced echelon form.

We keep row reducing the entire augmented matrix

$$\begin{bmatrix} 1 & 2 & 3 & 1 & 0 & 0 \\ 0 & 1 & 5 & 2 & 0 & -1 \\ 0 & 0 & 18 & 7 & 1 & -5 \end{bmatrix} \to \begin{bmatrix} 1 & 2 & 3 & 1 & 0 & 0 \\ 0 & 1 & 5 & 2 & 0 & -1 \\ 0 & 0 & 1 & 7/18 & 1/18 & -5/18 \end{bmatrix}$$

$$\begin{bmatrix} 1 & 2 & 3 & 1 & 0 & 0 \\ 0 & 1 & 5 & 2 & 0 & -1 \\ 0 & 0 & 1 & \frac{7}{18} & \frac{1}{18} & \frac{-5}{18} \end{bmatrix} \to \begin{bmatrix} 1 & 2 & 0 & -3/18 & -3/18 & 15/18 \\ 0 & 1 & 0 & 1/18 & -5/18 & 7/18 \\ 0 & 0 & 1 & 7/18 & 1/18 & -5/18 \end{bmatrix}$$

$$\begin{bmatrix} 1 & 2 & 0 & -\frac{3}{18} & \frac{-3}{18} & \frac{15}{18} \\ 0 & 1 & 0 & \frac{1}{18} & \frac{-5}{18} & \frac{-13}{9} \\ 0 & 0 & 1 & \frac{7}{18} & \frac{1}{18} & \frac{-5}{18} \end{bmatrix} \to \begin{bmatrix} 1 & 0 & 0 & -5/18 & 7/8 & 1/18 \\ 0 & 1 & 0 & 1/18 & -5/18 & 7/18 \\ 0 & 0 & 1 & 7/18 & 1/18 & -5/18 \end{bmatrix}$$

This is in reduced echelon form. The right half of the augmented matrix is the inverse of A.

Inverse

We computed that

$$\begin{bmatrix} 1 & 2 & 3 \\ 3 & 1 & 2 \\ 2 & 3 & 1 \end{bmatrix}^{-1} = \begin{bmatrix} -5/18 & 7/18 & 1/18 \\ 1/18 & -5/18 & 7/18 \\ 7/18 & 1/18 & -5/18 \end{bmatrix}.$$

Inverse

We computed that

$$\begin{bmatrix} 1 & 2 & 3 \\ 3 & 1 & 2 \\ 2 & 3 & 1 \end{bmatrix}^{-1} = \begin{bmatrix} -5/18 & 7/18 & 1/18 \\ 1/18 & -5/18 & 7/18 \\ 7/18 & 1/18 & -5/18 \end{bmatrix}.$$

If you multiply the two matrices you get the identity

$$\begin{bmatrix} 1 & 2 & 3 \\ 3 & 1 & 2 \\ 2 & 3 & 1 \end{bmatrix} \begin{bmatrix} -5/18 & 7/18 & 1/18 \\ 1/18 & -5/18 & 7/18 \\ 7/18 & 1/18 & -5/18 \end{bmatrix} = \begin{bmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{bmatrix}.$$

When is a matrix invertible?

We have seen that a matrix is invertible if you can reduce it to the identity matrix via row operations. This description is not complete enough for our purposes–we don't always want to have to run the row reduction algorithm every time.

Theorem (The Invertible Matrix Theorem)

If A is an $n \times n$ matrix then the following are equivalent:

1. *A is invertible*

Theorem (The Invertible Matrix Theorem)

If A is an $n \times n$ matrix then the following are equivalent:

1. *A is invertible*
2. *A is row equivalent to I_n*

Theorem (The Invertible Matrix Theorem)

If A is an $n \times n$ matrix then the following are equivalent:

1. *A is invertible*
2. *A is row equivalent to I_n*
3. *A has n pivot columns*

Theorem (The Invertible Matrix Theorem)

If A is an $n \times n$ matrix then the following are equivalent:

1. *A is invertible*
2. *A is row equivalent to I_n*
3. *A has n pivot columns*
4. *the equation $A\mathbf{x} = \mathbf{0}$ has only the trivial solution*

Theorem (The Invertible Matrix Theorem)

If A is an $n \times n$ matrix then the following are equivalent:

1. *A is invertible*
2. *A is row equivalent to I_n*
3. *A has n pivot columns*
4. *the equation $A\mathbf{x} = \mathbf{0}$ has only the trivial solution*
5. *the columns of A form a linearly independent set*

Theorem (The Invertible Matrix Theorem)

If A is an $n \times n$ matrix then the following are equivalent:

1. *A is invertible*
2. *A is row equivalent to I_n*
3. *A has n pivot columns*
4. *the equation $A\mathbf{x} = \mathbf{0}$ has only the trivial solution*
5. *the columns of A form a linearly independent set*
6. *the linear transformation $\mathbf{x} \mapsto A\mathbf{x}$ is one-to-one*

Theorem (The Invertible Matrix Theorem)

If A is an $n \times n$ matrix then the following are equivalent:

1. *A is invertible*
2. *A is row equivalent to I_n*
3. *A has n pivot columns*
4. *the equation $A\mathbf{x} = \mathbf{0}$ has only the trivial solution*
5. *the columns of A form a linearly independent set*
6. *the linear transformation $\mathbf{x} \mapsto A\mathbf{x}$ is one-to-one*
7. *the equation $A\mathbf{x} = \mathbf{b}$ has at least one solution for each $\mathbf{b} \in \mathbb{R}^n$*

Theorem (The Invertible Matrix Theorem)

If A is an $n \times n$ matrix then the following are equivalent:

1. *A is invertible*
2. *A is row equivalent to I_n*
3. *A has n pivot columns*
4. *the equation $A\mathbf{x} = \mathbf{0}$ has only the trivial solution*
5. *the columns of A form a linearly independent set*
6. *the linear transformation $\mathbf{x} \mapsto A\mathbf{x}$ is one-to-one*
7. *the equation $A\mathbf{x} = \mathbf{b}$ has at least one solution for each $\mathbf{b} \in \mathbb{R}^n$*
8. *the columns of A span $\mathbb{R}^n$*

Theorem (The Invertible Matrix Theorem)

If A is an $n \times n$ matrix then the following are equivalent:

1. *A is invertible*
2. *A is row equivalent to I_n*
3. *A has n pivot columns*
4. *the equation $A\mathbf{x} = \mathbf{0}$ has only the trivial solution*
5. *the columns of A form a linearly independent set*
6. *the linear transformation $\mathbf{x} \mapsto A\mathbf{x}$ is one-to-one*
7. *the equation $A\mathbf{x} = \mathbf{b}$ has at least one solution for each $\mathbf{b} \in \mathbb{R}^n$*
8. *the columns of A span $\mathbb{R}^n$*
9. *the linear transformation $\mathbf{x} \mapsto A\mathbf{x}$ maps $\mathbb{R}^n$ onto $\mathbb{R}^n$*

Theorem (The Invertible Matrix Theorem)

If A is an $n \times n$ matrix then the following are equivalent:

1. *A is invertible*
2. *A is row equivalent to I_n*
3. *A has n pivot columns*
4. *the equation $A\mathbf{x} = \mathbf{0}$ has only the trivial solution*
5. *the columns of A form a linearly independent set*
6. *the linear transformation $\mathbf{x} \mapsto A\mathbf{x}$ is one-to-one*
7. *the equation $A\mathbf{x} = \mathbf{b}$ has at least one solution for each $\mathbf{b} \in \mathbb{R}^n$*
8. *the columns of A span $\mathbb{R}^n$*
9. *the linear transformation $\mathbf{x} \mapsto A\mathbf{x}$ maps $\mathbb{R}^n$ onto $\mathbb{R}^n$*
10. *there is an $n \times n$ matrix C such that $CA = I_n$*

Theorem (The Invertible Matrix Theorem)

If A is an $n \times n$ matrix then the following are equivalent:

1. *A is invertible*
2. *A is row equivalent to I_n*
3. *A has n pivot columns*
4. *the equation $A\mathbf{x} = \mathbf{0}$ has only the trivial solution*
5. *the columns of A form a linearly independent set*
6. *the linear transformation $\mathbf{x} \mapsto A\mathbf{x}$ is one-to-one*
7. *the equation $A\mathbf{x} = \mathbf{b}$ has at least one solution for each $\mathbf{b} \in \mathbb{R}^n$*
8. *the columns of A span $\mathbb{R}^n$*
9. *the linear transformation $\mathbf{x} \mapsto A\mathbf{x}$ maps $\mathbb{R}^n$ onto $\mathbb{R}^n$*
10. *there is an $n \times n$ matrix C such that $CA = I_n$*
11. *there is an $n \times n$ matrix D such that $AD = I_n$*

Theorem (The Invertible Matrix Theorem)

If A is an $n \times n$ matrix then the following are equivalent:

1. *A is invertible*
2. *A is row equivalent to I_n*
3. *A has n pivot columns*
4. *the equation $A\mathbf{x} = \mathbf{0}$ has only the trivial solution*
5. *the columns of A form a linearly independent set*
6. *the linear transformation $\mathbf{x} \mapsto A\mathbf{x}$ is one-to-one*
7. *the equation $A\mathbf{x} = \mathbf{b}$ has at least one solution for each $\mathbf{b} \in \mathbb{R}^n$*
8. *the columns of A span $\mathbb{R}^n$*
9. *the linear transformation $\mathbf{x} \mapsto A\mathbf{x}$ maps $\mathbb{R}^n$ onto $\mathbb{R}^n$*
10. *there is an $n \times n$ matrix C such that $CA = I_n$*
11. *there is an $n \times n$ matrix D such that $AD = I_n$*
12. *A^T is invertible*

Proving the IMT: 1 implies 2 implies 3 implies 4 implies 5

If A is invertible then A is row equivalent to I_n by the previous section. So (1) implies (2).

Proving the IMT: 1 implies 2 implies 3 implies 4 implies 5

If A is invertible then A is row equivalent to I_n by the previous section. So (1) implies (2). If A is row-equivalent to I_n, then the row-operations that transform A to I_n actually transform A into reduced echelon form. Since I_n has no non-pivot columns, A has no non-pivot columns. So (2) implies (3)

If A is invertible then A is row equivalent to I_n by the previous section. So (1) implies (2). If A is row-equivalent to I_n, then the row-operations that transform A to I_n actually transform A into reduced echelon form. Since I_n has no non-pivot columns, A has no non-pivot columns. So (2) implies (3) Suppose A has n pivot columns. If we augment A with $\mathbf{0}$ and row-reduce, there are no free variables. Thus $A\mathbf{x} = \mathbf{0}$ has only the trivial solution. So (3) implies (4)

Proving the IMT: 1 implies 2 implies 3 implies 4 implies 5

If A is invertible then A is row equivalent to I_n by the previous section. So (1) implies (2). If A is row-equivalent to I_n, then the row-operations that transform A to I_n actually transform A into reduced echelon form. Since I_n has no non-pivot columns, A has no non-pivot columns. So (2) implies (3) Suppose A has n pivot columns. If we augment A with $\mathbf{0}$ and row-reduce, there are no free variables. Thus $A\mathbf{x} = \mathbf{0}$ has only the trivial solution. So (3) implies (4) Any linear dependence relation among the columns of A gives a non-trivial solution to $A\mathbf{x} = \mathbf{0}$. Thus (4) implies (5).

Application of the IMT

We can use the IMT to check if things are non-invertible really easily.

Application of the IMT

We can use the IMT to check if things are non-invertible really easily.

Example

Let $A = \begin{bmatrix} 1 & 0 & 1 & 2 \\ 2 & 0 & 2 & 4 \\ 3 & 0 & 2 & 6 \\ 4 & 1 & 3 & 8 \end{bmatrix}$.

Application of the IMT

We can use the IMT to check if things are non-invertible really easily.

Example

Let $A = \begin{bmatrix} 1 & 0 & 1 & 2 \\ 2 & 0 & 2 & 4 \\ 3 & 0 & 2 & 6 \\ 4 & 1 & 3 & 8 \end{bmatrix}$. Let's see if A is invertible without row reduction.

Application of the IMT

We can use the IMT to check if things are non-invertible really easily.

Example

Let $A = \begin{bmatrix} 1 & 0 & 1 & 2 \\ 2 & 0 & 2 & 4 \\ 3 & 0 & 2 & 6 \\ 4 & 1 & 3 & 8 \end{bmatrix}$. Let's see if A is invertible without row reduction. There's a whole bunch of ways to check, but one that is nice is: if A has linearly dependent columns, then A is not invertible.

Application of the IMT

We can use the IMT to check if things are non-invertible really easily.

Example

Let $A = \begin{bmatrix} 1 & 0 & 1 & 2 \\ 2 & 0 & 2 & 4 \\ 3 & 0 & 2 & 6 \\ 4 & 1 & 3 & 8 \end{bmatrix}$. Let's see if A is invertible without row reduction. There's a whole bunch of ways to check, but one that is nice is: if A has linearly dependent columns, then A is not invertible. We note that the fourth column of A is 2 times the first column of A.

Application of the IMT

We can use the IMT to check if things are non-invertible really easily.

Example

Let $A = \begin{bmatrix} 1 & 0 & 1 & 2 \\ 2 & 0 & 2 & 4 \\ 3 & 0 & 2 & 6 \\ 4 & 1 & 3 & 8 \end{bmatrix}$. Let's see if A is invertible without row reduction. There's a whole bunch of ways to check, but one that is nice is: if A has linearly dependent columns, then A is not invertible. We note that the fourth column of A is 2 times the first column of A. Thus the columns of A are linearly dependent, so A is not invertible by the IMT.

Invertible maps

Definition

A function $f : X \to Y$ is called **invertible** (or *bijective*) if there is a map $g : Y \to X$ such that $f(g(y)) = y$ for all $y \in Y$ and $g(f(x)) = x$ for all $x \in X$. If such a g exists, it is called the **inverse of** f.

Invertible maps

Definition

A function $f : X \to Y$ is called **invertible** (or *bijective*) if there is a map $g : Y \to X$ such that $f(g(y)) = y$ for all $y \in Y$ and $g(f(x)) = x$ for all $x \in X$. If such a g exists, it is called the **inverse of** f.

When is a linear transformation T given by $\mathbf{x} \mapsto A\mathbf{x}$ invertible?

Invertible maps

Definition

A function $f : X \to Y$ is called **invertible** (or *bijective*) if there is a map $g : Y \to X$ such that $f(g(y)) = y$ for all $y \in Y$ and $g(f(x)) = x$ for all $x \in X$. If such a g exists, it is called the **inverse of** f.

When is a linear transformation T given by $\mathbf{x} \mapsto A\mathbf{x}$ invertible? Precisely when A is an invertible matrix.

Theorem

Let A be $n \times n$ and let T be the linear transformation $\mathbf{x} \mapsto A\mathbf{x}$.

Theorem

Let A be $n \times n$ and let T be the linear transformation $\mathbf{x} \mapsto A\mathbf{x}$. Then T is invertible if and only if A is invertible.

Theorem

Let A be $n \times n$ and let T be the linear transformation $\mathbf{x} \mapsto A\mathbf{x}$. Then T is invertible if and only if A is invertible.

Proof.

Suppose that A is invertible. Define $S : \mathbb{R}^n \to \mathbb{R}^n$ by $S(\mathbf{x}) = A^{-1}\mathbf{x}$.

Theorem

Let A be $n \times n$ and let T be the linear transformation $\mathbf{x} \mapsto A\mathbf{x}$. Then T is invertible if and only if A is invertible.

Proof.

Suppose that A is invertible. Define $S : \mathbb{R}^n \to \mathbb{R}^n$ by $S(\mathbf{x}) = A^{-1}\mathbf{x}$. Then $T(S(\mathbf{x})) = AA^{-1}(\mathbf{x}) = \mathbf{x}$ and $S(T(\mathbf{x})) = \mathbf{x}$ for all $\mathbf{x} \in \mathbb{R}^n$.

Theorem

Let A be $n \times n$ and let T be the linear transformation $\mathbf{x} \mapsto A\mathbf{x}$. Then T is invertible if and only if A is invertible.

Proof.

Suppose that A is invertible. Define $S : \mathbb{R}^n \to \mathbb{R}^n$ by $S(\mathbf{x}) = A^{-1}\mathbf{x}$. Then $T(S(\mathbf{x})) = AA^{-1}(\mathbf{x}) = \mathbf{x}$ and $S(T(\mathbf{x})) = \mathbf{x}$ for all $\mathbf{x} \in \mathbb{R}^n$.
Suppose that T is invertible with inverse S. We show that $A\mathbf{x} = \mathbf{0}$ has only the trivial solution.

Theorem

Let A be $n \times n$ and let T be the linear transformation $\mathbf{x} \mapsto A\mathbf{x}$. Then T is invertible if and only if A is invertible.

Proof.

Suppose that A is invertible. Define $S : \mathbb{R}^n \to \mathbb{R}^n$ by $S(\mathbf{x}) = A^{-1}\mathbf{x}$. Then $T(S(\mathbf{x})) = AA^{-1}(\mathbf{x}) = \mathbf{x}$ and $S(T(\mathbf{x})) = \mathbf{x}$ for all $\mathbf{x} \in \mathbb{R}^n$. Suppose that T is invertible with inverse S. We show that $A\mathbf{x} = \mathbf{0}$ has only the trivial solution. If $T(\mathbf{x}) = T(\mathbf{0})$ then $S(T(\mathbf{x})) = S(T(\mathbf{0}))$. But S is the inverse of T, so $\mathbf{x} = \mathbf{0}$. Thus the only solution to $A\mathbf{x} = \mathbf{0}$ is the trivial solution. The IMT tells us that A is invertible. □