Answers to Even-Numbered Suggested Problems

6.2 #26. Nonzero orthogonal vectors are linearly independent, so $\dim W = n$ and hence it must be the full space $\mathbb{R}^n$.

6.2 #30. Orthogonality is not a property that depends on ordering: if the columns of U are orthogonal that is simply a statement about them as a set, and as a set they are the same as the columns of V.

6.3 #24. a. Every w vector is orthogonal to every other w vector by assumption that the basis is orthogonal; likewise for the v vectors. Every w vector is orthogonal to every v vector because the w vectors lie in W and the v's in $W^\perp$, and every vector in W is orthogonal to every vector in $W^\perp$.

b. Every vector in $\mathbb{R}^n$ may be written as a sum of a vector in W (its projection onto W) and a vector in $W^\perp$ (the orthogonal component), and hence as a linear combination of the bases of W and $W^\perp$ in the set from part (a).

c. In part (a) we showed the union of orthogonal bases for W and $W^\perp$ is orthogonal; this means it is also linearly independent. In part (b) we showed it spans $\mathbb{R}^n$. Therefore it is a basis for $\mathbb{R}^n$ and so contains n vectors, but it is also the union of sets of size $\dim W$ and $\dim W^\perp$, so those sum to n.

6.5 #20. This problem does not actually rely on the fact that you are multiplying A with its own transpose. Suppose A's columns are linearly dependent, so $c_1\boldsymbol{a}_1 + \ldots + c_n\boldsymbol{a}_n = \mathbf{0}$ for some set of scalars c_i not all zero. The columns of A^TA are $A^T\boldsymbol{a}_1, \ldots, A^T\boldsymbol{n}$, and

$$\begin{aligned} c_1A^T\boldsymbol{a}_1 + \ldots + c_nA^T\boldsymbol{a}_n &= A^Tc_1\boldsymbol{a}_1 + \ldots + A^Tc_n\boldsymbol{a}_n \\ &= A^T(c_1\boldsymbol{a}_1 + \ldots + c_n\boldsymbol{a}_n) \\ &= A^T\mathbf{0} = \mathbf{0} \end{aligned}$$

Hence A^TA's columns are also linearly dependent, and A^TA is not invertible.