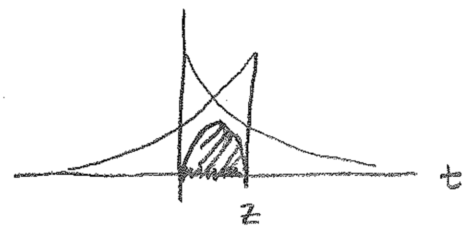
LECTURE # 14 JULY 27, 2015

HAND-OUT

① SUPPOSE X_1 AND X_2 ARE RVS W/ EXPONENTIAL DENSITIES AND PARAMETERS λ AND μ, RESPECTIVELY. FIND THE DENSITY FUNCTION FOR X_1 AND X_2.

$f_{X_1+X_2}(z) =$

$$\int_0^{\infty} \lambda e^{-\lambda t} f_{X_2}(z-t)\, dt$$



$$f_{X_2}(z-t) = \begin{cases} \mu e^{-\mu(z-t)}, & \text{IF } z-t>0 \ (\text{OR } z>t) \\ 0, & \text{IF } z-t<0 \ (\text{OR } z<t) \end{cases}$$

IF z IS NEGATIVE, THEN THE ABOVE IS ZERO. FOR z POSSITIVE, WE MUST HAVE $z>t$ SO

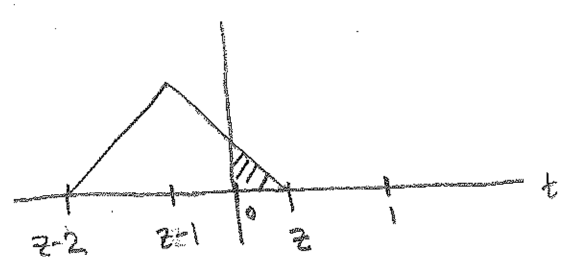
$$f_{X_1+X_2}(z) = \lambda\mu \int_0^z e^{-\lambda t} e^{-\mu(z-t)}\, dt = \lambda\mu e^{-\mu z} \int_0^z e^{t(\mu-\lambda)}\, dt = \frac{\lambda\mu e^{-\mu z}}{\mu-\lambda}\left(e^{z(\mu-\lambda)} - 1\right) = \frac{\lambda\mu}{\mu-\lambda}\left(e^{-z\lambda} - e^{-\mu z}\right)$$

② FROM EXAMPLE 14.2, COMPUTE THE DENSITY FUNCTION FOR THE CHANGE IN STOCK PRICE AFTER 3 DAYS OF TRADING.

NEED TO CONVOLVE $f_{X_1+X_2}$ AND f_{X_3}. EASIEST TO COMPUTE $f_{X_3} * f_{X_1+X_2}$

(AS OPPOSED TO $f_{X_1+X_2} * f_{X_3}$).

$$\int_0^1 f_{X_1+X_2}(z-t)\, dt$$



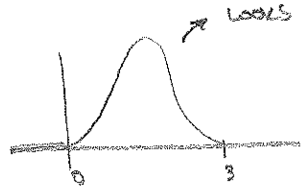
$$f_{X_1+X_2}(z-t) = \begin{cases} 0 & \text{IF } z<t \\ z-t & \text{IF } 0<z-t<1 \qquad z>t>z-1 \\ 2-z+t & \text{IF } 1<z-t<2 \qquad z-1>t>z-2 \\ 0 & \text{IF } z-t>2 \qquad t<z-2 \end{cases}$$

$$0 < z \le 1 : \quad \int_0^z (z-t)\,dt = \frac{z^2}{2}$$

$$1 < z \le 2 : \quad \int_0^{z-1} 2-z+t\,dt + \int_{z-1}^{1} z-t\,dt = -\frac{3}{2} + 3z - z^2$$

$$2 < z \le 3 : \quad \int_{z-2}^{1} 2-z+t\,dt = \frac{9}{2} - 3z + \frac{z^2}{2}$$

$$z > 3 : \quad 0$$



LOOKS LIKE A NORMAL DENSITY.

(3) THE <u>SUPPORT</u> OF A FUNCTION IS THE SET ON WHICH IT IS NON-ZERO. FOR DENSITY FUNCTIONS, THIS IS THE SAME AS:

$$\operatorname{supp}(f) = \{ x \in \mathbb{R} ;\ f(x) > 0 \}.$$

IF X AND Y ARE CTS RVs w/ DENSITY FUNCTIONS $f_X(t)$ AND $f_Y(t)$ RESPECTIVELY w/ SUPPORTS $[a,b]$ AND $[c,d]$, FIND THE SUPPORT OF $f_{X+Y}(z)$.

INTUITIVELY, IT SHOULD BE $[a+c, b+d]$ (JUST THINKING ABOUT THE POSSIBLE VALUES FOR X AND Y). HERE'S HOW TO SEE THIS:

$$f_{X+Y}(z) = \int_a^b f_X(t) f_Y(z-t)\,dt \neq 0 \quad \text{WHEN} \quad f_Y(z-t) \neq 0 \quad \text{(WHY?)}$$

THIS OCCURS PRECISELY WHEN $c < z-t < d$ OR $c+t < z < d+t$.

THIS IS TRUE FOR EVERY $t \in [a,b]$, SO $f_{X+Y}(z) > 0$ WHENEVER

$$c+a < z < d+b.$$

LECTURE #16
HAND OUT

JULY 31, 2015

(1) SUPPOSE A CHANGE IN A COMPANY'S STOCK PRICE ON A GIVEN DAY IS MODELED AS A SEQUENCE OF CTS RVs $X_1, X_2, \ldots$ THAT HAVE IDENTICAL DENSITY FUNCTIONS W/ $E(X_i) = 0$ AND $V(X_i) = 1$. ASSUME MOREOVER THAT THE DENSITY FUNCTION IS EVEN. WHAT CAN YOU SAY ABOUT THE PROB THAT THE STOCK'S PRICE WILL EXCEED \$105 AFTER 10 DAYS.

BY CHEBYSHEV'S INEQUALITY

$$P(|X_1 + \cdots + X_{10}| \geq 105) \leq \frac{1}{105^2}.$$ WE WANT TO LOOK AT

$P(X_1 + \cdots + X_{10} \geq 105)$, BUT $|X_1 + \cdots + X_{10}| \geq 105$ MEANS EITHER

$X_1 + \cdots + X_{10} \geq 105$ OR $(X_1 + \cdots + X_{10}) \leq -105$. THIS HAPPENS W/ EQUAL PROBABILITY. SO

$$P(X_1 + \cdots + X_{10} \geq 105) = \frac{1}{2} P(|X_1 + \cdots + X_{10}| \geq 105) \leq \frac{1}{2 \cdot 105^2}$$

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(2) ASSUME $X_1, \ldots, X_n$ ARE INDEPENDENT W/ POSSIBLY DIFFERENT DISTRIBUTIONS (OR DENSITIES IN THE CTS CASE). SET $m_k = E(X_k)$ AND $\sigma_k^2 = V(X_k)$. LET $M = m_1 + m_2 + \cdots + m_n$. ASSUME $\sigma_k^2 < R$ FOR ALL k. SHOW THAT FOR ANY $\varepsilon > 0$

$$P\left(\left| \frac{X_1 + \cdots + X_n}{n} - \frac{M}{n} \right| < \varepsilon \right) \to 1$$

AS $n \to \infty$.

HERE WE REDO THE ARGUMENT FOR THE LAW OF LARGE #s: $E\left(\frac{X_1 + \cdots + X_n}{n}\right) = \frac{M}{n}$ AND

$V\left(\frac{X_1 + \cdots + X_n}{n}\right) \leq \frac{nR}{n^2} = \frac{R}{n}$. SO USING CHEBYSHEV'S INEQUALITY WE HAVE:

$$0 \leq P\left(\left| \frac{X_1 + \cdots + X_n}{n} - \frac{M}{n} \right| < \varepsilon \right) \leq \frac{V\left(\frac{X_1 + \cdots + X_n}{n}\right)}{\varepsilon^2} \leq \frac{R}{n\varepsilon^2} \to 0 \text{ AS } n \to \infty.$$

LECTURE #17

HAND OUT

① THE IDEAL SIZE OF A FIRST YEAR CLASS AT A PARTICULAR COLLEGE IS 150. IT IS KNOWN FROM PAST EXPERIENCE THAT 30% ACCEPTED ACTUALLY ATTEND. USING THIS POLICY 450 STUDENTS ARE ACCEPTED. APPROXIMATELY WHAT IS THE PROB THAT MORE THAN 150 STUDENTS ATTEND THE COLLEGE?

$$P(X \geq 150.5) = \int_{a^*}^{\infty} \phi(x)\, dx, \qquad a^* = \frac{150.5 - 450 \cdot 0.3}{\sqrt{450 \cdot 0.3 \cdot 0.7}}$$

$$= \int_{1.59}^{\infty} \phi(x)\, dx \qquad\qquad = 1.594$$

$$= 0.0559$$

② A CLUB BASKETBALL TEAM WILL PLAY A 44-GAME SEASON. 26 GAMES AGAINST CLASS A TEAMS AND 18 AGAINST CLASS B TEAMS. SUPPOSE THE TEAM WILL WIN EACH GAME AGAINST A CLASS A TEAM W/ PROB 0.4, AND WIN EACH GAME AGAINST A CLASS B TEAM W/ PROB 0.7. (a) FIND THE APPROXIMATE PROBABILITY THAT THE TEAM WINS 25 OR MORE GAMES?

$X_A + X_B = Z$ IS A RV W/ MEAN 23 AND VARIANCE 10.02. WE WANT TO

COMPUTE $$P(X_A + X_B \geq 25) = P(X_A + X_B \geq 24.5)$$

$$= \int_{a^*}^{\infty} \phi(x)\, dx \qquad a^* = \frac{24.5 - 23}{\sqrt{10.02}} = 0.474$$

$$= \int_{0.474}^{\infty} \phi(x)\, dx \approx 0.3178.$$

(b) ASSUME THAT THE DIFFERENCE IN TWO BINOMIAL DISTRIBUTIONS IS BINOMIAL. APPROXIMATELY WHAT IS THE PROBABILITY THAT THE TEAM WINS MORE GAMES AGAINST CLASS A TEAMS THAN CLASS B TEAMS.

USE $$a = \frac{0.5 + 2.2}{\sqrt{10.02}} = 0.8529$$

INTEGRATE $$\int_{0.8529}^{\infty} \phi(x)\, dx = 0.1968$$

(3) THE AIM OF THIS PROBLEM IS TO GIVE SOME PLAUSIBILITY TO:

$$\lim_{n\to\infty} \sqrt{npq}\; P\left(\langle np + x\sqrt{npq}\rangle \text{ successes}\right) = \phi(x)$$

IN THE SPECIAL CASE $x = 0$ AND WE MAKE A SUBSEQUENCE OF ns s.t. np IS ALWAYS AN INTEGER.

(a) WRITE OUT $\sqrt{npq}\; P(np \text{ successes})$ IN TERMS OF FACTORIALS (NO BINOMIAL COEFFICIENTS).

$$P(np \text{ successes}) = \binom{n}{np} p^{np} q^{nq} = \frac{n!}{(np)!\,(nq)!}\, p^{np} q^{np} \cdot \sqrt{npq}$$

(b) USING THE FACT THAT $n! \sim \sqrt{2\pi n}\, n^n e^{-n}$ (STIRLING'S APPROX) SHOW

$$\sqrt{npq}\; P(np \text{ successes}) \sim \frac{1}{\sqrt{2\pi}}.$$

BY ABOVE WE HAVE $$\sqrt{npq}\; P(np \text{ successes}) \sim \frac{\sqrt{2\pi n}\, \cancel{n^n} \cancel{e^{-n}}\, \cancel{p^{np}}\, q^{np} \cdot \sqrt{npq}}{\sqrt{2\pi np}\, \sqrt{2\pi nq}\, (np)^{np} (nq)^{nq}\, \cancel{e^{-np}}\, \cancel{e^{-nq}}}$$

$$= \frac{1}{\sqrt{2\pi}}. \checkmark$$

LECTURE # 19

HAND OUT

(1) COLLEGES OFTEN QUOTE CERTAIN STATISTICS FOR FINANCIAL AID APPLICATIONS. FOR INSTANCE THEY MIGHT SAY THE AVG STUDENT HOUSING EXPENSE IS \$650 PER MONTH. SUPPOSE YOU SURVEY 175 STUDENTS AND FIND A SAMPLE MEAN OF \$616.91 SPENT FOR STUDENT HOUSING AND A SAMPLE STANDARD DEVIATION OF \$128.65. COMPUTE THE 95% CONFIDENCE INTERVAL FOR YOUR MEASUREMENT. IS THE SCHOOL'S QUOTE LIKELY TO REPRESENT THE ACTUAL STATISTIC?

WE NEED TO FIND WHEN $P\left(\left|616.91-\mu\right|<\varepsilon\right) \approx \int_{-2}^{2} \phi(x)\,dx$.

SO WE NEED TO SET :

$$\frac{\varepsilon\sqrt{175}}{128.65} = 2$$

SO $\varepsilon = 19.45$. THE 95% CONFIDENCE INTERVAL IS $(597.46, 636.36)$. SINCE THE SCHOOL'S QUOTE IS NOT IN THIS INTERVAL, IT IS LIKELY THAT THE STATISTIC IS INCORRECT.

(2) SUPPOSE YOU ROLL A DIE UNTIL THE SUM IS 300. WHAT IS THE APPROX PROB THAT AT LEAST 100 ROLLS ARE NECESSARY?

X_i = OUTCOME OF i^{th} DIE ROLL. WE WANT TO FIND WHEN

$$P(X_1+\dots+X_{100} < 300) \approx \int_{-\infty}^{299.5^*} \phi(x)\,dx = \int_{-\infty}^{-2.95} \phi(x)\,dx = 0.002$$

$$299.5^* = \frac{299.5-350}{10\times\sqrt{2.917}}$$

$E(X_1+\dots+X_{100}) = 3.5\times 100$

$V(X_1+\dots+X_{100}) = 100\times 2.917$

$$= -2.95$$

(3) LET X_i = THE OUTCOME OF A DIE ROLL. WHAT IS THE APPROX PROBABILITY THAT

$$e^6 \le X_1\cdot X_2\cdot\ldots\cdot X_{10} \le e^{10}\ ?$$

TAKING THE LOG OF BOTH SIDES YIELDS:

$$6 \leq \log(X_1 \cdots X_{10}) \leq 10 \quad \text{OR} \quad 6 \leq \log(X_1) + \cdots + \log(X_{10}) \leq 10$$

NOW $E(\log(X_i)) = \frac{1}{6} \cdot \sum_{i=1}^{6} \log(i) = 1.097$ AND

$V(\log(X_i)) = 0.366$, SO:

$$P\left(6 \leq \log(X_1) + \cdots + \log(X_{10}) \leq 10\right) \approx \int_{5.5^*}^{10.5^*} \phi(x)\,dx = \int_{-2.86}^{-0.25} \phi(x)\,dx$$

$$10.5^* = \frac{10.5 - 10 \times 1.097}{\sqrt{10} \times \sqrt{0.366}} = -0.25 \qquad = \boxed{0.399}$$

$$5.5^* = \frac{5.5 - 10 \times 1.097}{\sqrt{10}\sqrt{0.366}} = -2.859$$

(4) USE THE CENTRAL LIMIT THEOREM TO PROVE THE LAW OF LARGE #S.

SET $E(X_i) = \mu$ AND $V(X_i) = \sigma^2$ w/ $X_1, X_2, \ldots$ IDENTICAL INDEPENDENT RVs

$$P\left(\left|\frac{X_1 + \cdots + X_n}{n} - \mu\right| < \varepsilon\right) = P\left(\left|\left(\frac{X_1 + \cdots + X_n}{n} - \mu\right) \cdot \frac{\sqrt{n}}{\sigma}\right| < \frac{\sqrt{n}}{\sigma}\varepsilon\right)$$

$$\approx \int_{-\frac{\sqrt{n}\varepsilon}{\sigma}}^{\frac{\sqrt{n}\varepsilon}{\sigma}} \phi(x)\,dx$$

BY THE CENTRAL LIMIT THEOREM

AS $n \to \infty$ THE LAST INTEGRAL GOES TO 1. THUS

$$\lim_{n \to \infty} P\left(\left|\frac{X_1 + \cdots + X_n}{n} - \mu\right| < \varepsilon\right) = 1.$$