1) If today's stock costs Y_0 dollars,

the price in 10 days will be $Y_0 + \underbrace{X_1 + X_2 + \dots + X_{10}}_{\text{change for next 10 days}}$

Let $S_{10} = X_1 + \dots + X_{10}$.

As the X_i are independent normal r.v.'s

w/ mean 0, var 1, we see S_{10} is normal w/

$$E(S_{10}) = 0, \quad V(S_{10}) = 10 \cdot 1 = 10$$

Then for $Y_0 = 100$,

$$P(Y_{10} \geq 105) = P(Y_0 + S_{10} \geq 105)$$

$$= P(100 + S_{10} \geq 105)$$

$$= P(S_{10} \geq 5)$$

standardize $\rightarrow$ $= P\left(\frac{S_{10}}{\sqrt{10}} \geq 5/\sqrt{10}\right)$

$\left(\frac{S_{10}}{\sqrt{10}} \text{ is std normal}\right)$ $\approx \frac{1}{2} - NA(1.58) \approx \frac{1}{2} - .4429$

$= .0571$

2) We wish to construct a 99% confidence interval.

Consulting the normal table, we observe $NA(2.58) = .4951$,

so $2NA(2.58) = .9902$.

We then seek n such that $P(|\bar{p}_n - p| \leq .005) \geq .99$

where : $p \doteq$ proportion of females

$\bar{p}_n = S_n/n$ is our estimate of p.

As $V(\bar{p}_n) = pq/n$ and our 99% confidence interval

ranges from -2.58σ to 2.58σ,

we see that we need $p \in (\bar{p} - 2.58\sqrt{pq/n}, \bar{p} + 2.58\sqrt{\frac{pq}{n}})$

so we must have

$$2.58\sqrt{\frac{pq}{n}} \leq .005$$

Note $\sqrt{pq} \leq 1/2$, so this becomes

$$\frac{2.58}{\sqrt{n}} \leq .01,$$

i.e. $258 \leq \sqrt{n}$ $\Rightarrow$ $n \geq 258^2 = 66{,}564$

3) a) Let $S_{1000} = \#$ heads. There are two ways we can fail:

① The coin is fair & $S_{1000} \geq 525$

② The coin is biased & $S_{1000} < 525$

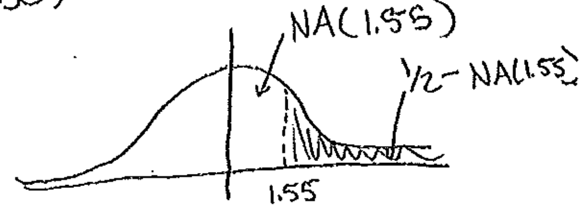
Case 1: Coin is fair: $E(S_{1000}) = .5(1000) = 500$

$V(S_{1000}) = (.5)(.5)1000 = 250$

$P(S_{1000} \geq 525) = P(S_{1000} \geq 524.5)$ (move to continuous)

$= P\left(\frac{S_{1000}-500}{\sqrt{250}} \geq \frac{24.5}{\sqrt{250}}\right)$ (standardize)

(normal approx. using CLT) $\rightarrow \approx \frac{1}{2} - NA(1.55)$



$\approx \frac{1}{2} - .4394 = .0616$

Case 2: Coin is biased: $E(S_{1000}) = (.55)(1000) = 550$

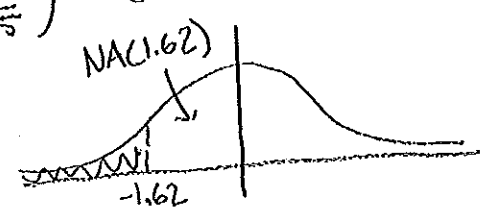
$V(S_{1000}) = (.55)(.45)(1000)$

$= 247.5$

$P(S_{1000} < 525) = P(S_{1000} < 524.5)$ (move to cts)

$= P\left(\frac{S_{1000}-550}{\sqrt{247.5}} < \frac{-25.5}{\sqrt{247.5}}\right)$ (standardize)

normal approx. $\rightarrow \approx \frac{1}{2} - NA(1.62)$



$\approx .0526$

So $P(\text{guess wrong}) = \underbrace{\frac{1}{2}(.0616)}_{\text{fair coin}} + \underbrace{\frac{1}{2}(.0526)}_{\text{biased coin}} = .0571.$

b) Using guess & check (in R), see 525 is optimal,

4) Let X_i represent the rolls of dice.

we compute

$$P(X_1 \cdot X_2 \cdot \ldots \cdot X_{100} \leq a^{100}) = P(\log(X_1 \cdot X_2 \cdot \ldots \cdot X_{100}) \leq \log(a^{100}))$$

$$= P(\log(X_1) + \ldots + \log(X_{100}) \leq 100 \log(a))$$

We can now use the central limit theorem to approximate this probability w/

$$E(\log X_i) = \sum_{i=1}^{6} \log(i)/6 = 1.0965 = \mu$$

(here log is natural log, or ln)

$$V(\log X_i) = E((\log X_i - \mu)^2) = .4391 = \sigma^2$$

so for $S_n = \log X_1 + \ldots + \log X_n$,

$E(S_n) = n\mu$, $V(S_n) = n\sigma^2$.

($\log_{10}$ gives $E(\log X_i) = .4762$, $V(\log X_i) = .0828$)

Then

$$P(S_{100} \leq 100 \log(a)) \underset{\text{standardize}}{=} P\left(\frac{S_{100} - 100\mu}{\sqrt{100\sigma^2}} \leq \frac{100 \log(a) - 100\mu}{\sqrt{100\sigma^2}}\right)$$

$$\approx P\left(Z \leq \frac{100 \log(a) - 109.6542}{6.6264}\right)$$

$$= F_Z\left(\frac{100 \log(a) - 109.6542}{6.6264}\right)$$

w/ Z the standard normal r.v. & F_Z its cdf

(for $\log_{10}$, have $F_Z\left(\frac{100 \log(a) - 47.6222}{2.8778}\right)$)

5) Assume each customer is using the phone w/ prob. $\approx 1/30$ during peak hours

(as some have pointed out, this is a kind of fishy assumption, but here a necessary one)

Then we have a Bernoulli trial process w/ $p = 1/30$, $n \doteq 2000$. We seek K s.t.

for $S_{2000} = \#$ successes, (phone users)

$$P(S_{2000} \leq K) \leq .01$$

We have
$$P(S_{2000} \leq K) \underset{\text{standardize}}{=} P\left(\frac{S_{2000} - \overbrace{2000/30}^{E(S_{2000})}}{\underbrace{\sqrt{2000\left(\frac{1}{30}\right)\left(\frac{29}{30}\right)}}_{SD(S_{2000})}} \leq \frac{K - 2000/30}{\sqrt{2000\left(\frac{1}{30}\right)\left(\frac{29}{30}\right)}} \right)$$

$$= P\left(S^*_{2000} \leq \frac{K - 66\,2/3}{8.0277} \right)$$

Consulting the normal table, we see we need 2.34 standard deviations to reach the desired likelihood,

so $\dfrac{K - 66\,2/3}{8.0277} \geq 2.34 \iff K \geq 2.34(8.0277) + 66\,2/3$
≥ 85.4516

so $K = 86$ (it must be an integer)

Using ~~poisscdf($\overset{\lambda}{200/3}$, 86) in MATLAB, we see the same value suffices.~~ ppois$(86, \frac{200}{3})$ we see same value suffices

6) We will prove for $X_1, X_2, \ldots$ ind. identically dist. r.v.'s w/ finite mean, variance and $S_n = X_1 + \ldots + X_n$ that for any $\varepsilon > 0$,

$$P\left(\left|\frac{S_n - \mu}{n}\right| \leq \varepsilon\right) \to 1 \quad \text{as } n \to \infty$$

(this is one statement of the law of large numbers)

Pf: Let $\varepsilon > 0$. Then for $\sigma = SD(X_i)$, we have

$$P\left(\left|\frac{S_n - \mu}{n}\right| \leq \varepsilon\right) = P\left(-\varepsilon \leq \frac{S_n - \mu}{n} \leq \varepsilon\right)$$

$$= P\left(-\frac{\varepsilon\sqrt{n}}{\sigma} \leq \frac{S_n - \mu}{\sqrt{n}\,\sigma} \leq \frac{\varepsilon\sqrt{n}}{\sigma}\right)$$

$$= P\left(-\frac{\varepsilon\sqrt{n}}{\sigma} \leq S_n^* \leq \frac{\varepsilon\sqrt{n}}{\sigma}\right)$$

as $n \to \infty$, this goes to $\displaystyle\int_{-\frac{\varepsilon\sqrt{n}}{\sigma}}^{\frac{\varepsilon\sqrt{n}}{\sigma}} \frac{1}{\sqrt{2\pi}} e^{-x^2/2}\,dx$

by the central Limit Theorem.

Since $\frac{\varepsilon\sqrt{n}}{\sigma} \to \infty$ & $-\frac{\varepsilon\sqrt{n}}{\sigma} \to -\infty$ as $n \to \infty$,

this integral approaches $\displaystyle\int_{-\infty}^{\infty} \frac{1}{\sqrt{2\pi}} e^{-x^2/2}\,dx = 1.$

$\square$