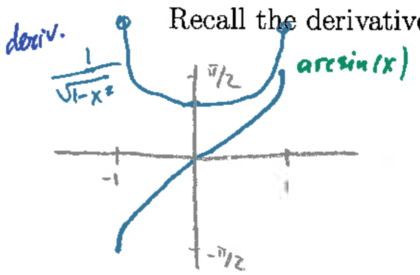
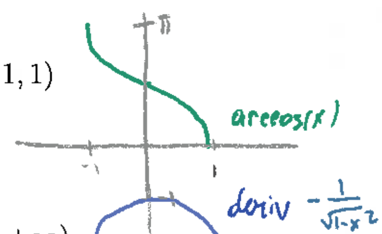
1. MORE DERIVATIVES (20 MINS)

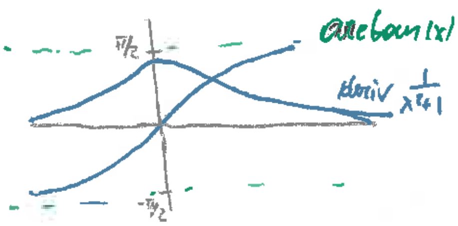
Recall the derivatives



$$\frac{d(\arcsin(x))}{dx} = \frac{1}{\sqrt{1-x^2}}, \text{domain is } (-1,1)$$

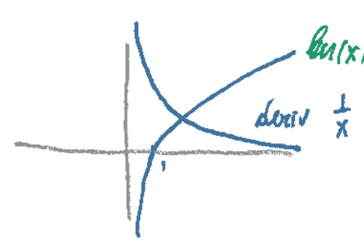
$$\frac{d(\arccos(x))}{dx} = \frac{-1}{\sqrt{1-x^2}} \text{domain is } (-1,1)$$





$$\frac{d(\arctan(x))}{dx} = \frac{1}{1+x^2} \text{domain is } (-\infty,+\infty)$$

$$\frac{d\log_a(x)}{dx} = \frac{1}{\ln(a)x}, \quad \text{for } a>0, \text{domain is } (0,+\infty)$$

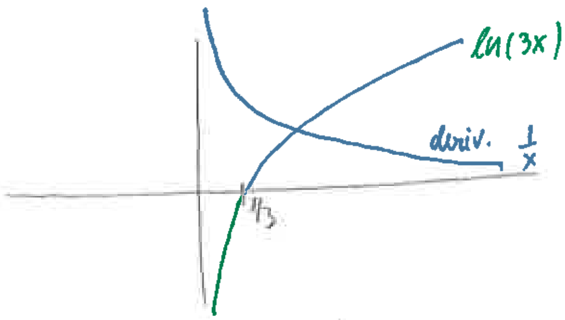


Since the derivative gives slopes of the tangent lines, and we only have tangent lines at points in the domain of the initial function, the domain of the derivative is contained in the domain of the initial function. (Draw graphs for each of the functions above and their derivative)

Exercises: Find the derivatives of the following functions and their domains:

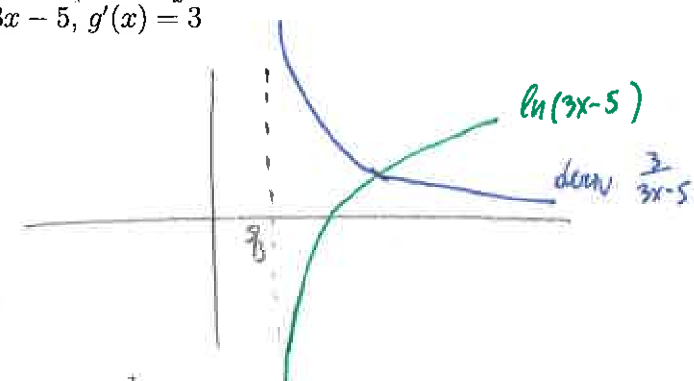
(1) $\frac{d}{dx}(\ln(3x)) = \frac{1}{3x} \cdot 3 = \frac{1}{x}$ by the chain rule, domain of the derivative is $(0, +\infty)$

$f(x) = \ln(x), f'(x) = \frac{1}{x}$
$g(x) = 3x,\ g'(x) = 3$
Also, $\ln(3x) = \ln(3) + \ln(x)$, so $\frac{d}{dx}(\ln(3x)) = \frac{d}{dx}(\ln(3) + \ln(x)) = 0 + \frac{1}{x}$.

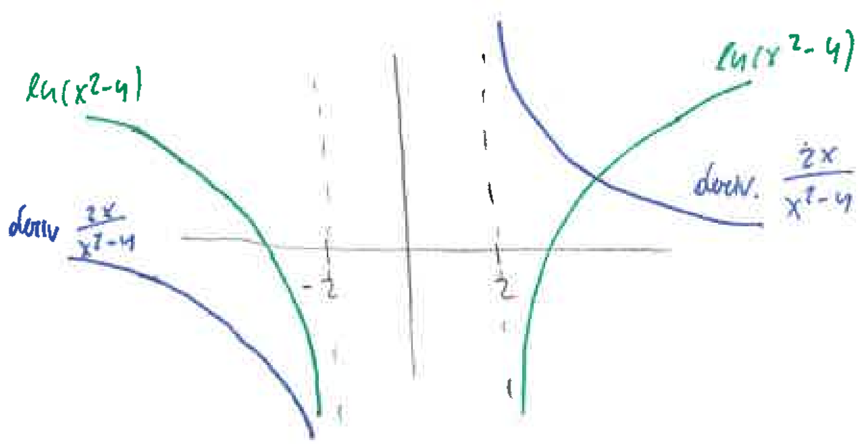


(2) $\frac{d}{dx}(\ln(3x-5)) = \frac{1}{3x-5} \cdot 3 = \frac{3}{3x-5}$, domain of the derivative is $(5/3, +\infty)$

$f(x) = \ln(x), f'(x) = \frac{1}{x}$
$g(x) = 3x - 5,\ g'(x) = 3$

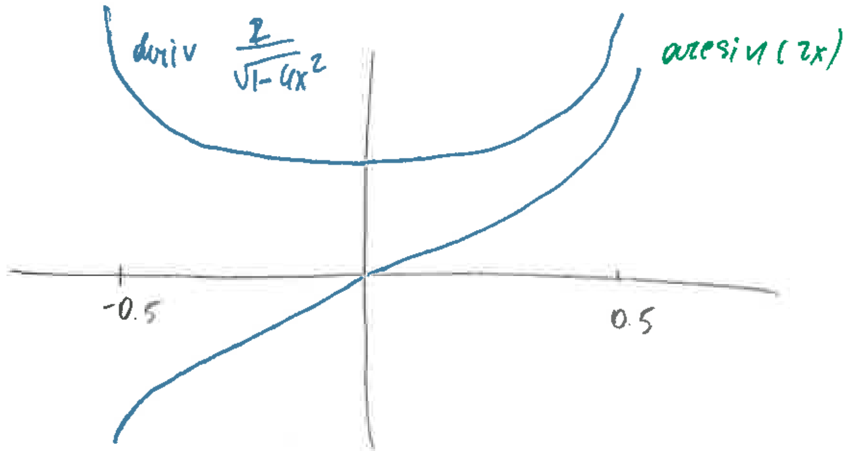


(3) $\frac{d}{dx}(\ln(x^2-4)) = \frac{1}{x^2-4} \cdot 2x$, domain of the derivative is $(-\infty, -2) \cup (2, +\infty)$
$f(x) = \ln(x), f'(x) = \frac{1}{x}$
$g(x) = x^2 - 4,\ g'(x) = 2x$



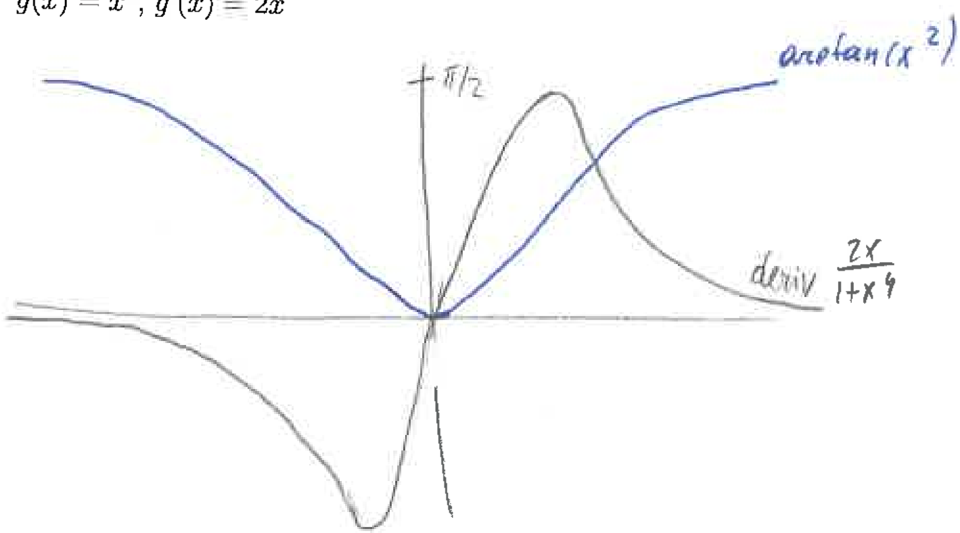
(4) $\frac{d}{dx}(\arcsin(2x)) = \frac{1}{\sqrt{1-(2x)^2}} \cdot 2$, domain of the derivative is $(-1/2, 1/2)$

$f(x) = \arcsin(x)$, $f'(x) = \frac{1}{\sqrt{1-x^2}}$
$g(x) = 2x$, $g'(x) = 2$



(5) $\frac{d}{dx}(\arctan(x^2)) = \frac{1}{1+(x^2)^2} \cdot 2x = \frac{2x}{1+x^4}$, domain of the derivative is $(-\infty, +\infty)$

$f(x) = \arctan(x)$, $f'(x) = \frac{1}{1+x^2}$
$g(x) = x^2$, $g'(x) = 2x$



(1.10)

2. Practice quiz (10 mins) if time permits

Exercise 1. Find $\frac{dy}{dx}$ for $y^3 = x^2 + y$, and the equation of the tangent line at the points $(1,0)$ and $(0,1)$

Solution: We differentiate with respect to x:

$\frac{d}{dx}(y^3) = \frac{d}{dx}(x^2 + y)$

$\frac{dy^3}{dy}\frac{dy}{dx} = \frac{d}{dx}(x^2) + \frac{d}{dx}y$ The LHS follows from the chain rule, the RHS from the sum rule

$3y^2\frac{dy}{dx} = 2x + \frac{dy}{dx}$

$3y^2\frac{dy}{dx} - \frac{dy}{dx} = 2x$

$(3y^2 - 1)\frac{dy}{dx} = 2x$

$\frac{dy}{dx} = \frac{2x}{3y^2 - 1}$

The point $(1,0)$ is not on the curve: $0^3 \neq 1^2 + 0$, so there is no tangent line. But the point $(0,1)$ is in the curve: $1^3 = 0^2 + 1$, and then the slope is $\frac{2\cdot 0}{3\cdot 1^2 - 1} = 0$, so the equation of the tangent line is $y - 1 = 0(x - 0)$, or $y = 1$.

Exercise 1. Find the derivative of $\ln(2x+1)$, and the domain of the derivative.

Solution: First, the domain of $\ln(2x+1)$ is given by $2x + 1 > 0$, so the domain of the logarithm is $(-0.5, +\infty)$. Thus the derivative can only be defined on this interval.

To find the equation of the derivative, e use the chain rule: let $f(x) = \ln(x), g(x) = 2x+1$, then $f'(x) = \frac{1}{x}, g'(x) = 2$, and

$$(\ln(2x+1))' = \frac{1}{2x+1} \cdot 2 = \frac{2}{2x+1}.$$

Exercise: Find the following derivatives:

$$(\arccos(e^x))' = \frac{-1}{\sqrt{1-(e^x)^2}} \cdot e^x$$

$$(\ln(\arctan(x)))' = \frac{1}{\arctan(x)} \cdot \frac{1}{1+x^2}$$

$$(x\ln(x))' = (x)'\ln(x) + x(\ln(x))' = \ln(x) + x \cdot \frac{1}{x} = \ln(x) + 1$$

(1.30)

3. Newton's method (rest of time)

Motivation. Suppose tht a car dealer offers to sell you a car for \$18,000, or for payments of \$375 per month for five years. Which option should you take?

Just multiplying $375 * 5 * 12 = 22,500$ won't tell us the whole story: there is a thing such as inflation and interest rates, so \$18,000 today will not be worth the same in five years. Instead, we need to find out what interest rate would the dealer be charging if we choose the second option.

Luckily, there is a formula to calculate the future value of an investment or annuity given a rate of return or an interest rate, and other related formulas. In our case, we want a way to calculate the interest rate given the present value of an annuity and the value of the monthly payemnts. Then

$$A = \frac{R}{i}[1 - (1+i)^{-n}]$$

where A is the present value of the annuity, R the value of each payment, n the number of payments, and i the interest rate. Then we have

$$18000 = \frac{375}{i}[1 - (1+i)^{-60}]$$

and we want to solve for i. Then

$$18000 = \frac{375}{i}\frac{(1+i)^{60} - 1}{(1+i)^{60}}$$

$$18000i(1+i)^{60} = 375([(1+i)^{60} - 1]$$

$$48i(1+i)^{60} = (1+i)^{60} - 1$$

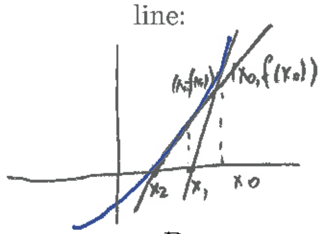
Replacing i by x, we have

$$48x(1+x)^{60} - (1+x)^{60} + 1 = 0$$

This is a 61-degree polynomial. There is no formula we can use to find roots of such large degree polynomials akin to the quadratic formla. Instead, we try to approvimate the value. Plugging this into a calculator, it will tell us that the positive root is $x \approx 0.0076$. As a monthly iterest rate, this is rather large, and it would be more profitable to just buy the car on the spot.

How does the calculator find the roots, if it doesn't employ formulas? Let's look at the following picture: we have some polynomial, and we want to approximate the root. We start with an approximation x_0. How may we use tangent lines to get better approximations?

Let's explain algebraically what happened here. We started with an approximation x_0, and we drew the tangent line at $(x_0, f(x_0))$. Its equation is $y - f(x_0) = f'(x_0)(x - x_0)$. The x-intercept is given by $(x_1, 0)$, and we can solve for x_1 if $f'(x_0) \neq 0$ given the equation of the line:



$$0 - f(x_0) = f'(x_0)(x_1 - x_0)$$
$$x_1 - x_0 = -\frac{f(x_0)}{f'(x_0)}$$
$$x_1 = x_0 - \frac{f(x_0)}{f'(x_0)}$$

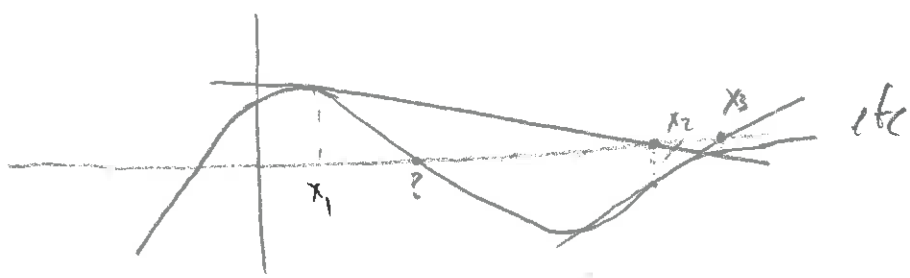
But we saw that we were able to get a better approximation by doing the same again. iven the new approximation, x_1, the point on the graph is $(x_1, f(x_1))$, then going through the same process again we get a new approximation

$$x_2 = x_1 - \frac{f(x_1)}{f'(x_1)}$$

In general, if we keep iterating, given an approximation x_n we get

$$x_{n+1} = x_n - \frac{f(x_n)}{f'(x_n)}.$$

Does this always work? Erm, no, not if the first approximation was a bad one. What do we mean by bad? We mean that the derivative was very small. The equation above doesn't work if $f'(x_0) = 0$, but it also doesn't work that well when $f'(x_0)$ is close to 0.



Let's practice! **Exercise:** Estimate $\sqrt[6]{2}$ correct to eight decimal places.

Solution: $\sqrt[6]{2}$ is a solution of the equation $x^6 - 2$. So let $f(x) = x^6 - 2$, then $f'(x) = 6x^5$.

We know that $\sqrt[6]{2}$ is somewhere between 1 and 2, so let the first approximation be $x_0 = 1$. Since $f'(1) = 6$ is not close to 0, we may proceed. Then the general formula for the approximations will be

$$x_{n+1} = x_n - \frac{x_n^2 - 2}{2x_n}$$

So

$$x_1 = 1 - \frac{1^6 - 2}{6 \cdot 1^5} = 1 - \frac{-1}{6} = \frac{7}{6} \approx 1.16666667$$
$$x_2 = \frac{7}{6} - \frac{(\frac{7}{6})^6 - 2}{6 \cdot (\frac{7}{6})^5} \approx 1.12644368$$

If we keep doing so,

$$x_3 \approx 1.12249707$$
$$x_4 \approx 1.12246205$$
$$x_5 \approx 1.22246205$$

Since x_4 and x_5 agree to eight decimal places, a good approximation should be

$$\sqrt[6]{2} \approx 1.22246205.$$