1. More implicit differentiation (20 mins)

See attached page

2. Derivatives of inverse trig functions (20 mins)

Implicit differentiation is quite helpful in finding derivatives of various types of inverse functions: today we will tackle inverse trig functions, and tomorrow logarithmic functions.

Derivative of $\arcsin(x)$ Let $y = \arcsin(x)$. Then $\sin(y) = \sin(\arcsin(x)) = x$ and $\frac{-\pi}{2} \leq y \leq \frac{\pi}{2}$. Differentiating implicitly $\sin(y) = x$ w.r.t. x, we have $\cos(y)\frac{dy}{dx} = 1$ so

$$\frac{dy}{dx} = \frac{1}{\cos(y)}$$

Since $-\pi/2 \leq y \leq \pi/2$, $\cos(y)$ is positive, and since we have the equality $\sin^2(y)+\cos^2(y) = 1$, then $\cos(y) = \sqrt{1-\sin^2(y)} = \sqrt{1-x^2}$ since $\sin(y) = x$. Then

$$\frac{d(\arcsin(x))}{dx} = \frac{1}{\sqrt{1-x^2}}, \qquad \text{for } -1 < x < 1$$

Derivative of $\arccos(x)$ **(optional, if time permits)** Let $y = \arccos(x)$. Then $\cos(y) = \cos(\arccos(x)) = x$ and $0 \leq y \leq \pi$. Differentiating implicitly $\cos(y) = x$ w.r.t. x, we have $-\sin(y)\frac{dy}{dx} = 1$ so

$$\frac{dy}{dx} = \frac{-1}{\sin(y)}$$

Since $0 \le y \le \pi$, $\sin(y)$ is positive, and since we have the equality $\sin^2(y) + \cos^2(y) = 1$, then $\sin(y) = \sqrt{1 - \cos^2(y)} = \sqrt{1 - x^2}$ since $\cos(y) = x$. Then

$$\frac{d \arccos(x)}{dx} = \frac{-1}{\sqrt{1-x^2}}, \qquad \text{for } -1 < x < 1$$

Derivative of $\arctan(x)$ First, we prove $\sec^2(x) = 1 + \tan^2(x)$:

$$1 + \tan^2(x) = 1 + \frac{\sin^2(x)}{\cos^2(x)} = \frac{\cos^2(x) + \sin^2(x)}{\cos^2(x)} = \frac{1}{\cos^2(x)} = \sec^2(x)$$

Now we proceed to find $\frac{d}{dx}\arctan(x)$: let $y = \arctan(x)$, then $\tan(y) = \tan(\arctan(x)) = x$ for all real x. Now we differentiate $\tan(y) = x$ implicitly:

$$\frac{d}{dx}\tan(y) = \frac{d}{dx}x$$

$$\sec^2(y)\frac{dy}{dx} = 1$$

$$\frac{dy}{dx} = \frac{1}{\sec^2(y)} = \frac{1}{1 + \tan^2(y)} = \frac{1}{1+x^2}$$

So the derivative formula is

$$\frac{d}{dx}\arctan(x) = \frac{1}{1+x^2}$$

Exercises:

(1) $\frac{d}{dx}(x \arctan(x)) = \arctan(x) + x \cdot \frac{1}{1+x^2}$

(2) $\frac{d}{dx}(\arcsin(3x)) = \frac{1}{\sqrt{1-(3x)^2}} \cdot 3$

(3) $\frac{d}{dx}\left(\frac{\arccos(x)}{e^{2x}}\right) = \frac{-\frac{1}{\sqrt{1-x^2}}e^{2x} - \arccos(x)e^{2x}2}{(e^{2x})^2}$

(1.35)

3. Derivatives of logarithmic functions (rest of time)

We know the derivative of e^x, what about $\frac{d}{dx}2^x$? Can write $2^x = e^{\ln(2^x)} = e^{x\ln(2)}$. Then

$$\frac{d}{dx}2^x = \frac{d}{dx}e^{x\ln(2)} = e^{x\ln(2)}\ln(2) = e^{\ln(2^x)}\ln(2) = 2^x\ln(2)$$

Works for any $a > 0$:

$$\frac{d}{dx}a^x = \ln(a)a^x$$

Derivative of $\ln(x)$ Let $y = \ln(x)$, $x > 0$. Then $e^y = e^{\ln(x)} = x$. Differentiating implicitly w.r.t. x, we have $e^y \frac{dy}{dx} = 1$ so

$$\frac{dy}{dx} = \frac{1}{e^y} = \frac{1}{x}$$

$$\frac{d\ln(x)}{dx} = \frac{1}{x}, \qquad \text{for } x > 0$$

Although $\frac{1}{x}$ is defined for $x \neq 0$, the derivative $(\ln(x))' = \frac{1}{x}$ is only defined for $x > 0$, since we only have tangent lines where $\ln(x)$ is defined.

Derivative of $\log_a(x)$ Let $y = \log_a(x)$, $x > 0, a > 0$. Then $a^y = a^{\log_a(x)} = x$. Differentiating implicitly w.r.t. x, we have $\ln(a)a^y \frac{dy}{dx} = 1$ so

$$\frac{dy}{dx} = \frac{1}{\ln(a)a^y} = \frac{1}{\ln(a)x}$$

$$\frac{d\log_a(x)}{dx} = \frac{1}{\ln(a)x}, \qquad \text{for } x > 0, a > 0$$

Exercises:

(1) $\frac{d}{dx}(x\ln(x)) = \ln(x) + x \cdot \frac{1}{x} = \ln(x) + 1$

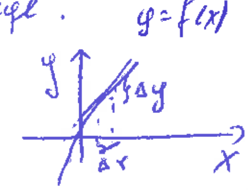
(2) $\frac{d}{dx}(\ln(2x)) = \frac{2}{2x} = \frac{1}{x}$

(3) $\frac{d}{dx}(\log_3(x^2)) = \frac{2x}{\ln(3)(x^2)} = \frac{2}{\ln(3)x}$

(4) $\frac{d}{dx}\left(\frac{x}{\ln(x)}\right) = \frac{\ln(x) - 1}{\ln(x)^2}$

talk about the derivative. We started the derivative thinking of it as speed : given a small change in time, how much does the distance I run change?

Same with any function:



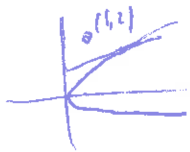
the derivative of $f(x)$ w.r.t. x $\frac{dy}{dx}$ is the tells you what change in y you have given a "small" change in x.

$x \xrightarrow{\frac{dy}{dx}} y \xrightarrow{\frac{dy^2}{dy}} y^2$ what is the total change $\frac{dy^2}{dx} = \frac{dy^2}{dy} \cdot \frac{dy}{dx} = 2y\frac{dy}{dx}$

$x \xrightarrow{\frac{dy}{dx}} y \xrightarrow{\frac{d\cos(y)}{dy}} \cos(y)$ $\quad \frac{d\cos(y)}{dx} = \frac{d\cos(y)}{dy} \cdot \frac{dy}{dx} = -\sin(y)\frac{dy}{dx}$

$x \xrightarrow{\frac{dy}{dx}} y \xrightarrow{\frac{dy^3}{dy}} y^3$: $\frac{dy^3}{dx} = \frac{dy^3}{dy} \cdot \frac{dy}{dx} = 3y^2\frac{dy}{dx}$.

~~$x \xrightarrow{\frac{dy}{dx}} y \xrightarrow{d} xy$~~



slope of tangent line at an arbitrary point (x,y)

Ex 1: Find $\frac{dy}{dx}$ for $y^2 = x$

$\frac{d}{dx} y^2 = \frac{d}{dx} x$

$\frac{dy^2}{dy} \cdot \frac{dy}{dx} = 1$

$2y\frac{dy}{dx} = 1$

$\frac{dy}{dx} = \frac{1}{2y}$ Not defined when points not on the curve: (e.g. (1,2)) b/c $2^2 \neq 1$ or $y = 0$ and $y^2 = x$ (so (0,0)) vertical.

Find $\frac{dx}{dy}$ $\quad \frac{d}{dy} y^2 = \frac{d}{dy} x$

$2y = \frac{dx}{dy}$; defined when $y^2 = x$.

Ex: Find $\frac{dy}{dx}$ for $x^3 + y^3 = 6xy$

$\frac{d}{dx}(x^3 + y^3) = \frac{d}{dx}(6xy)$

$\frac{d}{dx}x^3 + \frac{d}{dx}y^3 = 6\frac{d}{dx}(xy)$

$3x^2 + \frac{dy^3}{dy} \cdot \frac{dy}{dx} = 6\left(\frac{d(x)}{dx} \cdot y + x\frac{dy}{dx}\right)$

$3x^2 + 3y^2\frac{dy}{dx} = 6\left(y + x\frac{dy}{dx}\right)$

$3x^2 + 3y^2\frac{dy}{dx} = 6y + 6x\frac{dy}{dx}$

$3y^2\frac{dy}{dx} - 6x\frac{dy}{dx} = 6y - 3x^2$

$(3y^2 - 6x)\frac{dy}{dx} = 6y - 3x^2$ $\quad \to$ when (x,y) not on the curve.

$\frac{dy}{dx} = \frac{6y - 3x^2}{3y^2 - 6x}$ Not defined when $3y^2 - 6x = 0$ AND $x^3 + y^3 = 6xy$.

E.g. (0,0)