NAME: Key

MATH 1 MIDTERM 1

October 17, 2007

INSTRUCTIONS: This is a closed book, closed notes exam. You are not to provide or receive help from any outside source during the exam.

- Print your name clearly in the space provided.
- You may not use a calculator.

HONOR STATEMENT:

I have neither given nor received help on this exam, and all of the answers are my own.

Key

Signature

Question	Points	Score
1	9	9
2	12	12
3	4	4
4	6	6
5	10	10
6	10	10
7	8	8
8	10	10
9	11	11
10	20	20
Total:	100	100

1. Determine the inverse function $f^{-1}(x)$.

(a) [1 point] $f(x) = x^3$

$y = x^3$: swap x & y; $x = y^3$

cube roots : $\sqrt[3]{x} = y$.

$\boxed{f^{-1}(x) = \sqrt[3]{x}}$

(b) [1 point] $f(x) = 2^x$

$y = 2^x$: swap x & y; $x = 2^y$

$\log_2$: $\log_2 x = \log_2(2^y)$

power rule : $\log_2 x = y \log_2 2$

$\log_2 2 = 1$: $\log_2 x = y$.

$\boxed{f^{-1}(x) = \log_2 x}$

(c) [1 point] $f(x) = e^x$

Same as previous problem, but with base e instead of 2:

$f^{-1}(x) = \log_e x = \ln x$. $\boxed{f^{-1}(x) = \ln x}$

(d) [1 point] $f(x) = \log_3 x$

$y = \log_3 x \xrightarrow{\text{swap}} x = \log_3 y$, $\xrightarrow{\text{3 to power of each side:}} 3^x = 3^{\log_3 y}$

$3^{\log_3 y} = y$ (inverses), so $3^x = y$. $\boxed{f^{-1}(x) = 3^x}$

(e) [1 point] $f(x) = \tan x$

Inverse of tangent is arctan or $\tan^{-1}$.

$\boxed{f^{-1}(x) = \arctan x \quad \text{or} \quad \tan^{-1}(x)}$

(f) [2 points] $f(x) = -\frac{1}{x}$

$y = -\frac{1}{x} \xrightarrow{\text{swap}} x = -\frac{1}{y} \xrightarrow{\text{multiply } y} yx = -1 \xrightarrow{\text{divide } x} y = -\frac{1}{x}$.

$\boxed{f^{-1}(x) = -\frac{1}{x}}$ ← It's the same as $f(x)$!

(g) [2 points] $f(x) = \sqrt{2x - 1}$

$y = \sqrt{2x-1} \xrightarrow{\text{swap}} x = \sqrt{2y-1} \xrightarrow{\text{square}} x^2 = 2y - 1$

$\xrightarrow{\text{add 1}} x^2 + 1 = 2y \xrightarrow{\text{divide 2}} \frac{x^2+1}{2} = y$

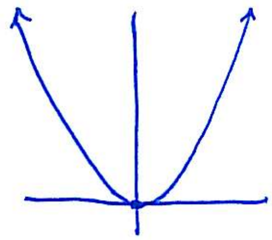
$\boxed{f^{-1}(x) = \frac{x^2+1}{2}}$

2. State the domain and range of the following functions.

(a) [2 points] $f(x) = x^2$

Domain: $(-\infty, \infty)$

Range: $[0, \infty)$

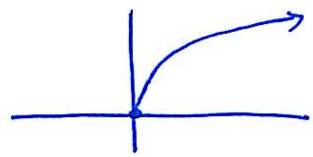


Can plug in any real number, out comes a nonnegative number.

(b) [2 points] $f(x) = \sqrt{x}$

Domain: $[0, \infty)$

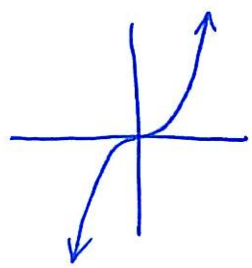
Range: $[0, \infty)$



(c) [2 points] $f(x) = x^3$

Domain: $(-\infty, \infty)$

Range: $(-\infty, \infty)$

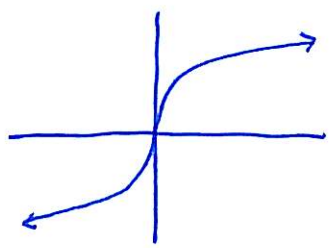


These are inverses of each other, so the domains and ranges swap. Can't really tell here, though...

(d) [2 points] $f(x) = \sqrt[3]{x}$

Domain: $(-\infty, \infty)$

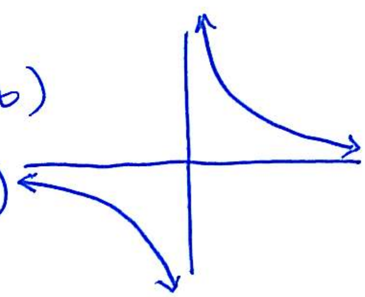
Range: $(-\infty, \infty)$



(e) [2 points] $f(x) = \frac{1}{x}$

Domain: $(-\infty, 0) \cup (0, \infty)$

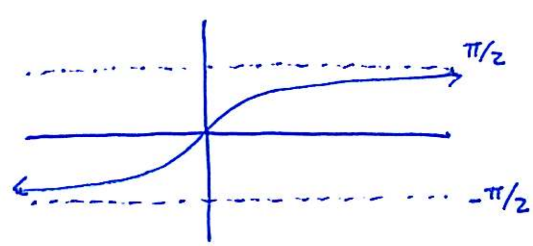
Range: $(-\infty, 0) \cup (0, \infty)$



(f) [2 points] $f(x) = \arctan x$

Domain: $(-\infty, \infty)$

Range: $(-\frac{\pi}{2}, \frac{\pi}{2})$



3. Let $f(x) = x^3 - 2x + 1$.

(a) [1 point] Compute $f(0)$.

$$f(0) = 0^3 - 2(0) + 1$$
$$= 0 - 0 + 1$$
$$= \boxed{1}$$

(b) [1 point] Compute $f(2)$.

$$f(2) = 2^3 - 2(2) + 1$$
$$= 8 - 4 + 1$$
$$= \boxed{5}$$

(c) [1 point] Find the slope of the line passing through the points $(0, f(0))$ and $(2, f(2))$.

$$m = \frac{f(2) - f(0)}{2 - 0} = \frac{5-1}{2} = \frac{4}{2} = \boxed{2}.$$

(d) [1 point] Find the equation of the line passing through the points $(0, f(0))$ and $(2, f(2))$.

$$m = \frac{y - f(0)}{x - 0}$$

$$2 = \frac{y-1}{x}, \quad 2x = y - 1, \quad \boxed{y = 2x + 1}.$$

4. Let $d(t) = t^2 - 1$ represent the distance an object has traveled in time t.

(a) [2 points] Determine the average velocity of the object in the interval $[1, 2]$.

The object starts at 1, ends at 2:

$$\text{average velocity} = \frac{d(2) - d(1)}{2-1} = \frac{(2^2-1) - (1^2-1)}{1}$$

$$= \frac{3-0}{1} = \boxed{3}$$

(b) [2 points] Evaluate and simplify $\frac{d(1+h)-d(1)}{h}$.

$$\frac{d(1+h) - d(1)}{h} = \frac{(1+h)^2 - 1 - (1^2-1)}{h}$$

$$= \frac{1 + 2h + h^2 - 1 - 0}{h}$$

$$= \frac{2h+h^2}{h} = \frac{\cancel{h}(2+h)}{\cancel{h}} \underset{h \neq 0}{=} \boxed{2+h}$$

(c) [2 points] The expression above represents the average velocity of the object in an interval $[1, 1+h]$. Plug in $h = 0.1, 0.01$, and 0.001 into the simplified form of the expression (or the complicated one if you prefer!) and estimate

$$\lim_{h \to 0} \frac{d(1+h) - d(1)}{h}.$$

$[1, 1+h]$	h	$2+h$
$[1, 1.1]$	0.1	2.1
$[1, 1.01]$	0.01	2.01
$[1, 1.001]$	0.001	2.001
		⋮

Looks like it goes to 2!

5. Starting with the function $y = \frac{1}{x}$, obtain $f(x)$ by taking $\frac{1}{x}$ and translating it right one unit followed by reflecting it about the x-axis. Obtain $g(x)$ by taking $\frac{1}{x}$ and reflecting it about the x-axis followed by translating it up one unit.

(a) [2 points] What is $f(x)$?

start with $\frac{1}{x}$

replace x by $x-1$: $\frac{1}{x-1}$

replace $f(x)$ by $-f(x)$: $-\frac{1}{x-1}$

$\boxed{f(x) = -\frac{1}{x-1}}$

(b) [2 points] What is $g(x)$?

Start with $\frac{1}{x}$

replace $f(x)$ by $-f(x)$: $-\frac{1}{x}$

replace $-f(x)$ by $-f(x)+1$: $-\frac{1}{x}+1$.

(i.e., just add 1)

$\boxed{g(x) = -\frac{1}{x}+1}$

(c) [2 points] Compute $f \circ g$.

$f \circ g(x) = f(g(x)) = -\frac{1}{g(x)-1} = -\frac{1}{(-\frac{1}{x}+1)-1}$

$= -\frac{1}{-\frac{1}{x}} = -1 \cdot (-x) = \boxed{x}$

(d) [2 points] Compute $g \circ f$.

$g \circ f(x) = g(f(x)) = -\frac{1}{f(x)} + 1 = -\frac{1}{-\frac{1}{x-1}} + 1$

$= -1 \cdot (-(x-1)) + 1 = \boxed{x}$

(e) [2 points] Given the results from parts (c) and (d), what relationship exists between f and g?

Since $f \circ g$ and $g \circ f$ both are x, f and g are inverse functions.

6. Simplify the following expressions:

(a) [2 points] $16^{-3/4}$

$$16^{-3/4} = \frac{1}{(\sqrt[4]{16})^3} \quad \left(\text{This is also } \frac{1}{\sqrt[4]{(16^3)}}, \text{ but I'd rather not do } 16^3 \text{ first!} \right)$$

$$= \frac{1}{2^3} = \boxed{\frac{1}{8}}$$

(b) [2 points] $\frac{x}{y} - \frac{y}{x}$

$$\frac{x}{y} - \frac{y}{x} = \frac{x}{y}\cdot\frac{x}{x} - \frac{y}{x}\cdot\frac{y}{y} \quad \text{(to get a common denominator)}$$

$$= \frac{x^2}{xy} - \frac{y^2}{xy} = \boxed{\frac{x^2-y^2}{xy}.}$$

(c) [2 points] $\log_8(64)$

$$\log_8 64 = \log_8(8^2) = 2\log_8 8 = 2\cdot 1 = \boxed{2.}$$

(d) [2 points] $\underbrace{\log_2(6) - \log_2(15)} + \log_2(20)$

$$\log_2\left(\frac{6}{15}\right) + \log_2 20 = \log_2\left(\frac{2}{5}\cdot 20\right)$$

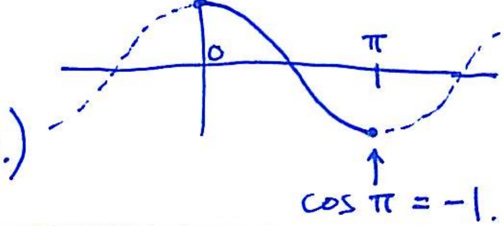
$$= \log_2 8$$

$$= \log_2(2^3) = 3\log_2 2 = 3\cdot 1 = \boxed{3}$$

(e) [2 points] $\arccos(-1)$

$\arccos(-1) = \theta \iff \cos\theta = -1$

$\arccos x$ has range $[0, \pi]$
(since we choose this domain of cos. to take the inverse of.)

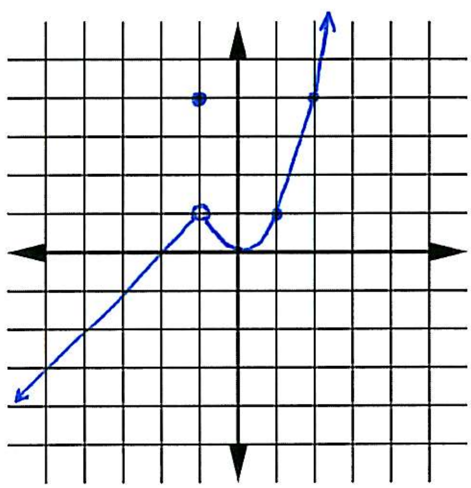


Since $\cos\pi = -1$, $\theta = \pi$ and $\boxed{\arccos(-1) = \pi}$.

7. Let

$$f(x) = \begin{cases} x+2 & \text{if } x < -1 \\ x^2 & \text{if } x > -1 \\ 4 & \text{if } x = -1 \end{cases}.$$

(a) [4 points] Graph $f(x)$.



(b) [1 point] Find $\lim_{x\to -1^-} f(x)$.

$\boxed{\lim_{x\to -1^-} f(x) = 1}$

(c) [1 point] Find $\lim_{x\to -1^+} f(x)$.

$\boxed{\lim_{x\to -1^+} f(x) = 1}$

(d) [1 point] Find $\lim_{x\to -1} f(x)$.

Since $\lim_{x\to -1^-} f(x)$ and $\lim_{x\to -1^+} f(x)$ are both 1, so is $\lim_{x\to -1} f(x)$.

$\boxed{\lim_{x\to -1} f(x) = 1}$

(e) [1 point] Find $f(-1)$.

$\boxed{f(-1) = 4.}$

8. Solve for x.

(a) [2 points] $x - 3 = 2 - \frac{x}{2}$

Multiply by 2 : $2x - 6 = 4 - x$

Add x : $3x - 6 = 4$

Add 6 : $3x = 10$

Divide 3 : $\boxed{x = \frac{10}{3}}$

(b) [2 points] $\left(\frac{1}{3}\right)^x = 27$

$(3^{-1})^x = 27$

$3^{-x} = 3^3$

Now take $\log_3$ on both sides:
$\log_3(3^{-x}) = \log_3(3^3)$

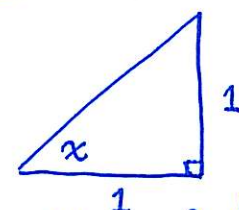
Power rule: $-x \log_3 3 = 3 \log_3 3$

$\log_3 3 = 1$: $-x = 3$

Mult. by -1 : $\boxed{x = -3}$

(c) [2 points] $\tan(x) = 1$ with x in the interval $(-\frac{\pi}{2}, \frac{\pi}{2})$

Draw a triangle with $\frac{\text{opp.}}{\text{adj.}} = 1$:



This is one of the "special triangles," with angle $\frac{\pi}{4}$. Since this is in our interval, $\boxed{x = \frac{\pi}{4}.}$

(d) [2 points] $\sin(\arcsin(x)) = 1$

sin and arcsin are inverse functions, so they cancel.

$\boxed{x = 1.}$

(e) [2 points] $\ln((x + 1)^3) = 3$

Power rule: $3 \ln(x+1) = 3$

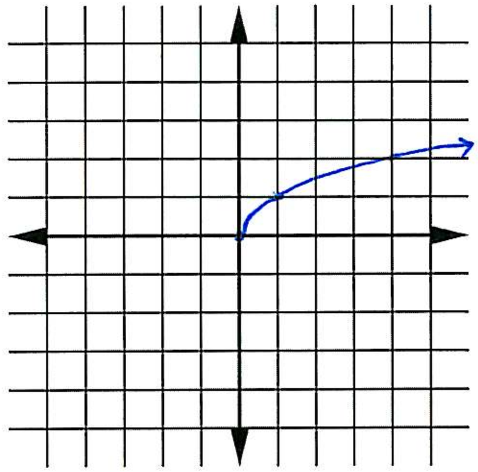
Divide by 3: $\ln(x+1) = 1$

Take e to both sides: $e^{\ln(x+1)} = e^1$

e and $\ln$ are inverses: $(x+1) = e$

Subtract 1 : $\boxed{x = e - 1.}$

9. [1 point] Sketch the graph of $f(x) = \sqrt{x}$.



Write the equations for the graphs that are obtained from the graph of $f(x)$ as follows:

(a) [2 points] Translate to the left by 3 units.

Replace x by $x+3$: $\boxed{\sqrt{x+3}}$

(b) [2 points] Stretch horizontally by a factor of 4.

Replace x by $\frac{x}{4}$: $\boxed{\sqrt{\frac{x}{4}}}$

(c) [2 points] Reflect about the y-axis.

Replace x by $-x$: $\boxed{\sqrt{-x}}$

(d) [2 points] First (a) then (c).

From (a) we have $\sqrt{x+3}$.

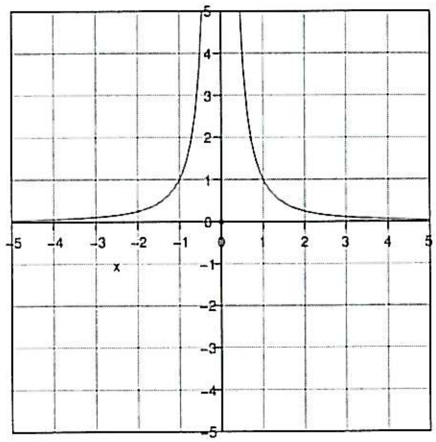
Replace x by $-x$: $\boxed{\sqrt{(-x)+3}}$

(e) [2 points] First (b) then (a).

From (b) we have: $\sqrt{\frac{x}{4}}$.

Replace x by $x+3$: $\boxed{\sqrt{\frac{x+3}{4}}}$

10. Consider the function $f(x) = \frac{1}{x^2}$ graphed below.



(a) [2 points] Find the domain of f.

$(-\infty, 0) \cup (0, \infty)$

(b) [2 points] Find the range of f.

$(0, \infty)$

(c) [1 point] Is f one-to-one?

No, it's easy to find two x-values with the same y-value, say $(-1,1)$ and $(1,1)$. (Also fails horiz. line test.)

(d) [1 point] What kind of symmetry does f have (even, odd, neither)?

Even. Fold the paper along the y-axis and see.

Better yet: Replace x by $-x$: $f(-x) = \frac{1}{(-x)^2} = \frac{1}{x^2}$, so $f(-x) = f(x)$.

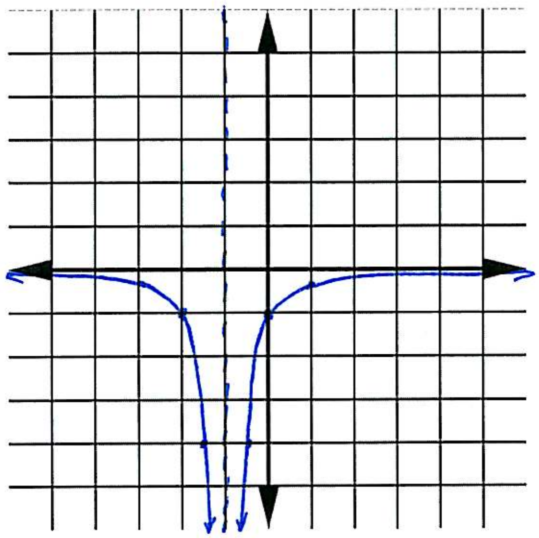
(e) [2 points] Determine the two transformations (in order) needed to obtain $g(x) = -\frac{1}{(x+1)^2}$ from $f(x)$.

Actually, in this case order doesn't matter.

Do one of these followed by the other:

reflect around x-axis, translate left by 1 unit.

(f) [1 point] Sketch the graph of $g(x)$.



(g) [2 points] Find the domain of g. $(-\infty,-1)\cup(-1,\infty)$ ← It's the domain of f shifted left 1 unit.

(h) [2 points] Find the range of g. $(-\infty,0)$ Negative of range of f.

(i) [1 point] Find $\lim_{x\to 0^+} g(x)$. -1

(j) [1 point] Find $\lim_{x\to 0^-} g(x)$. -1

(k) [1 point] Find $\lim_{x\to 0} g(x)$. -1 (since $\lim_{x\to 0^-} g(x)$ and $\lim_{x\to 0^+} g(x)$ are both -1.)

(l) [1 point] Find $g(0)$. -1

(m) [1 point] Find $\lim_{x\to -1^+} g(x)$. $-\infty$

(n) [1 point] Find $\lim_{x\to -1^-} g(x)$. $-\infty$

(o) [1 point] Find $\lim_{x\to -1} g(x)$. $-\infty$ (Again, $\lim_{x\to -1^+} g(x)$ and $\lim_{x\to -1^-} g(x)$ are the same and are both $-\infty$, so $\lim_{x\to -1} g(x) = -\infty$.)