Dartmouth College

Mathematics 17

Assignment 3

due Wednesday, January 25

In the first two problems, we consider two equivalence relations in addition to the one given in class by congruence modulo n. The first is the notion of fractions or rational numbers which we have mentioned as a motivation; the second will define the notion of the projective line.

1. Fractions. Let $S = \{(a, b) \in \mathbb{Z} \times \mathbb{Z} \mid b \neq 0\}$. Define a relation on S by saying $(a, b) \sim (c, d)$ if and only if $ad - bc = 0$.
   (a) Show that $\sim$ is an equivalence relation on S.
   (b) Following the model from class, we denote by $[(a, b)]$ the equivalence class containing (a, b), that is $[(a, b)] = \{(c, d) \in S \mid (c, d) \sim (a, b)\}$. For example, list five elements in $[(12, 15)]$. Of course, being sophisticated budding mathematicians, we recognize $[(a, b)]$ as nothing more that our friend the fraction a/b which can be expressed in many equivalent ways depending on the needs of the moment.

2. Let's start our study of projective space by considering the projective line (say) over $\mathbb{R}$. Let $S = \mathbb{R}^2 \setminus \{(0, 0)\} = \{(a, b) \in \mathbb{R}^2 \mid (a, b) \neq (0, 0)\}$, that is the whole plane except the origin. Define a relation on S by $(a, b) \sim (c, d)$ if and only if $(a, b) = \lambda(c, d) = (\lambda c, \lambda d)$ for some (necessarily) nonzero scalar $\lambda \in \mathbb{R}$.
   (a) Show that $\sim$ is an equivalence relation on S.
   (b) Denote the equivalence class of (a, b) by $[(a, b)]$ or for shorthand by $[a, b]$. Geometrically describe the points in a fixed equivalence class $[a, b]$.
   (c) Since $[a, b] = [\lambda a, \lambda b]$ for any nonzero λ, we have two cases, points where $b = 0$ and points where $b \neq 0$. If $b \neq 0$, show that $[a, b] = [a', 1]$ for some $a' \in \mathbb{R}$. Moreover show that $[a, 1] = [a', 1]$ if and only if $a = a'$, so the points on the projective line of the form $[a, 1]$ are in one-to-one correspondence with the points in $a \in \mathbb{R}$ (the affine line).
   (d) Show that there is just one point on the affine line $[a, b]$ with $b = 0$. This is the "point at infinity". This means the projective line is just a copy of the affine line together with an extra point at infinity, usually denoted $[1, 0]$.

3. Using Euclid's algorithm, compute the gcd of 12345 and 67890 and express it as a linear combination of 12345 and 67890 as in Bezout's theorem.

4. Find the general solution to $65x \equiv 75 \pmod{120}$.

5. Find the smallest number of marbles in a jar so that one remains if taken out 2, 3, 5 at a time, but none remain if taken out 11 at a time.