Dartmouth College
Mathematics 17

Assignment 1
due Wednesday, January 11

1. Describe all the rational points on $x^n + y^n = 1$ for $n > 2$, and justify your answer.

2. Consider the following set S of differentiable real-valued functions:

$$S = \{f \mid f'' = f\},$$

that is, functions which are solutions to the homogeneous differential equation $y'' - y = 0$.

   (a) Show that S is a vector space over $\mathbb{R}$.

   (b) Based on our work in class, what should the dimension of S be? Can you think of a calculus reason which makes this plausible?

   (c) Find an appropriate number of (independent) functions which lie in S, that is a basis for S.

   (d) Describe S in terms of this basis.

3. We said in class that two conics can intersect in 0, 1, 2, 3, or 4 points. Find explicit equations for pairs of examples of each type. For example, $x^2 + y^2 = 1$ and $(x-2)^2 + y^2 = 1$ intersect in (how many?) points.

4. Carefully write up a proof of Bachet's duplication formula that we considered in class: Consider the elliptic curve $y^2 = x^3 + k$ ($k \neq 0$). Show that if (a, b) ($b \neq 0$) is a point on the curve, then so is $\left(\dfrac{a^4 - 8ak}{4b^2}, \dfrac{-a^6 - 20a^3k + 8k^2}{8b^3}\right)$. This formula obviously makes no sense for $b = 0$. What is happening geometrically when $b = 0$?