A little linear algebra

See (for example) Linear Algebra and its Applications *by David Lay for a more thorough (and better) introduction.*

Suppose we have a pair of linear equations in 2 variables:

$$\begin{array}{rcrcr} 2x_1 & + & 3x_2 & = & 8 \\ x_1 & - & x_2 & = & -1 \end{array}$$

To solve this, we may cancel x_1 by subtracting twice the second equation from the first, arriving at:

$$\begin{array}{rcrcr} & & 5x_2 & = & 10 \\ x_1 & - & x_2 & = & -1 \end{array}$$

Next solve for x_2 by multiplying the first equation by $\frac{1}{5}$:

$$\begin{array}{rcrcr} & & x_2 & = & 2 \\ x_1 & - & x_2 & = & -1 \end{array}$$

Finally find x_1 by adding the first equation to the second:

$$\begin{array}{rcrcr} & & x_2 & = & 2 \\ x_1 & & & = & 1 \end{array}$$

Thus $(x_1, x_2) = (1, 2)$ is the unique solution to the original system.

Suppose instead we had 2 equations in 3 variables, such as:

$$\begin{array}{rcrcrcr} 2x_1 & + & 3x_2 & & & = & 8 \\ x_1 & - & x_2 & + & x_3 & = & -1 \end{array}$$

Performing the same sequence of operations as before, we arrive at:

$$\begin{array}{rcrcrcr} x_1 & & & + & \frac{3}{5}x_3 & = & 1 \\ & & x_2 & - & \frac{2}{5}x_3 & = & 2 \end{array}$$

Here we have infintely many possible solutions. We may choose x_3 freely, then these equations tell us what x_1 and x_2 must be. So, for example, choosing $x_3 = 0$ gives the solution $(x_1, x_2, x_3) = (1, 2, 0)$; choosing $x_3 = -1$ gives $(x_1, x_2, x_3) = (\frac{8}{5}, \frac{8}{5}, 1)$

Given a general system of m equations in n unknowns, which we will denote

$$\begin{array}{ccccccc} a_{11}x_1 & + & \dots & + & a_{1n}x_n & = & b_1 \\ \vdots & & & & \vdots & & \vdots \\ a_{m1}x_1 & + & \dots & + & a_{mn}x_n & = & b_m \end{array}$$

this same procedure, called *Gaussian elimination* allows us to find the solution(s) or deduce that the system has no solution. These actions are called *row operations* , which we record here:

- *Multiply an equation by a nonzero constant*
- *Add a multiple of an equation to another equation*
- *Swap two equations*

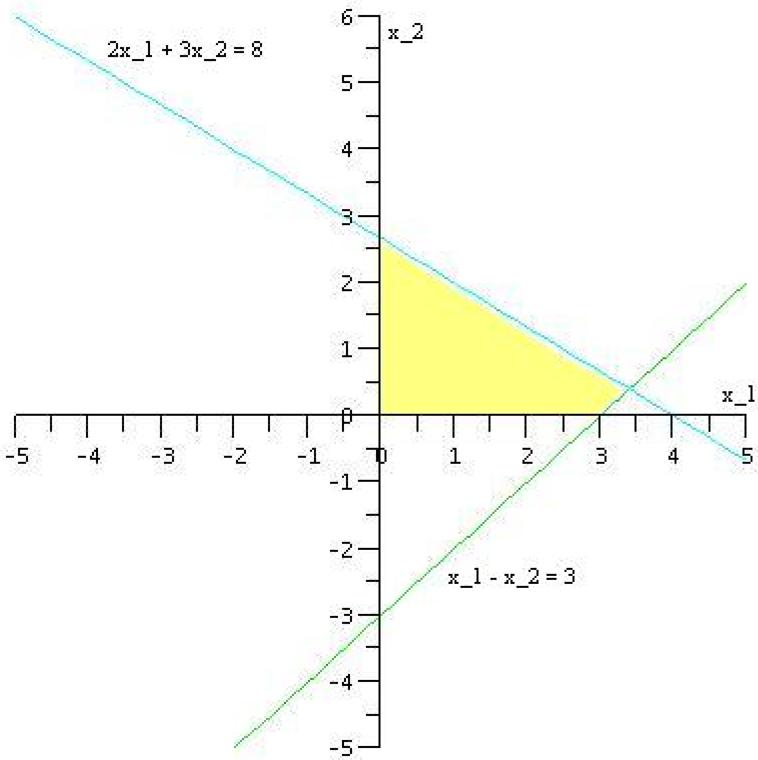
The key is that none of these operations will change the *feasible region* (or *solution set*) of the system. Each operation preserves the equalities of the original system, so solutions remain solutions. Since each of the operations is invertible, no new solutions are created, because these would yield new solutions to the original system.

The third operation may seem mysterious, even pointless. Indeed it will serve no purpose in our current setting, but will become useful when we turn to matrices later in the course. Let us agree for now that this is an allowable operation, i.e. one which will not change the feasible region.

As well as linear equalities we can consider systems of linear inequalities, for example

$$\begin{array}{ccccc} 2x_1 & + & 3x_2 & \leq & 8 \\ x_1 & - & x_2 & \leq & 3 \\ x_1 & & & \geq & 0 \\ & & x_2 & \geq & 0 \end{array}$$

We may draw the graph of this system; any solution must lie to the appropriate side of the lines formed by replacing each inequality by an equality. For example, the inequality $x_2 \geq 0$ says the solution must lie on or above the x_1-axis, that is the line $x_2 = 0$. Putting these together, we find that the feasible region is the shaded region indicated:



There is however a close connection between inequalities and equalities. Suppose we have the inequality $x_1 - 7x_2 \geq 3$. Rearranging, we have

$$x_1 - 7x_2 - 3 \geq 0$$

This difference is the *slack* in the inequality. We introduce a *slack variable* w_1, i.e. $w_1 = x_1 - 7x_2 - 3$. Then a system equivalent to ours is

$$\begin{array}{rcrcrcr} w_1 & = & x_1 & - & 7x_2 & - & 3 \\ w_1 & \geq & 0 & & & & \end{array}$$

Exercises

These are optional and not for credit. Use them at your discretion, and please ask me if you would like help with this material or feedback on your attempts at these problems.

1. Solve the following systems of equations:

(a)
$$\begin{array}{rcrcl} x_1 & + & 7x_2 & = & 9 \\ 3x_1 & - & x_2 & = & 5 \\ x_1 & + & x_2 & = & 3 \end{array}$$

(b)
$$\begin{array}{rcrcl} x_1 & + & 4x_2 & = & 6 \\ x_1 & - & 1 & = & x_2 \end{array}$$

(c)
$$\begin{array}{rcrcrcl} x_1 & + & 4x_2 & - & 3x_3 & = & 11 \\ 2x_1 & - & 2x_2 & + & x_3 & = & 1 \end{array}$$

2. Why in the first rule do we not allow multiplication by 0?

3. Suppose we have the following system:

$$\begin{array}{rcrcrcl} 2x_1 & + & x_2 & - & 4x_3 & = & 17 \\ x_1 & + & x_2 & - & x_3 & = & 1 \\ & & & & x_1 & \geq & 0 \\ & & & & x_2 & \geq & 0 \\ & & & & x_3 & \geq & 0 \end{array}$$

   How might we convert this into an equivalent system where all the equations are inequalities?