Chap. 2

9a. $\vec{V}(x,y,z) = (xz, yz, 0)$ S is the sphere $x^2+y^2+z^2=4$

For the top: $z = \sqrt{1-x^2-y^2}$ D is the circle of radius 4 in the xy-plane

$$d\vec{S} = \left(dx, 0, \tfrac{\partial z}{\partial x}\, dx\right) \times \left(0, dy, \tfrac{\partial z}{\partial y}\, dy\right)$$

$$= \left(1, 0, -x(1-x^2-y^2)^{-1/2}\right) \times \left(0, 1, -y(1-x^2-y^2)^{-1/2}\right) dx\,dy$$

$$= \left(x(1-x^2-y^2)^{-1/2}, -y(1-x^2-y^2)^{-1/2}, 1\right) dx\,dy$$

$$\Phi_{top} = \iint_D \vec{V}\cdot d\vec{S} = \iint_D \left(x(1-x^2-y^2)^{1/2}, y(1-x^2-y^2)^{1/2}, 0\right)\cdot\left(x(1-x^2-y^2)^{-1/2}, -y(1-x^2-y^2)^{-1/2}, 1\right) dx\,dy$$

$$= \iint_D (x^2 - y^2)\, dx\,dy \underset{\substack{\uparrow \\ \text{switch} \\ \text{to polar}}}{=} \iint_D r^2(\cos^2\theta - \sin^2\theta)\, r\,dr\,d\theta = \int_0^{2\pi}\int_0^4 r^3 \cos 2\theta\, dr\,d\theta$$

$$= 0$$

For the bottom: $z = -\sqrt{1-x^2-y^2}$ D is the circle of radius 4 in the xy-plane

$$d\vec{S} = \left(dx, 0, x(1-x^2-y^2)^{-1/2}dx\right) \times \left(0, dy, y(1-x^2-y^2)^{-1/2}dy\right)$$

$$= \left(-x(1-x^2-y^2)^{-1/2}, y(1-x^2-y^2)^{-1/2}, 1\right) dx\,dy$$

$$\Phi_{bottom} = \iint_D \left(-x(1-x^2-y^2)^{\frac{1}{2}}, -y(1-x^2-y^2)^{1/2}, 0\right)\cdot\left(-x(1-x^2-y^2)^{-1/2}, y(1-x^2-y^2)^{-1/2}, 1\right) dx\,dy$$

$$= \iint_D (x^2-y^2)\, dx\,dy = \iint_D r^2(\cos^2\theta - \sin^2\theta)\, r\,dr\,d\theta = 0$$

$$\Rightarrow \Phi_{tot} = 0$$

10 In the book.

11 a. $\vec{V}(x,y,z) = (2,3,-1)$ $\hat{N} = \left(\frac{x}{5}, \frac{y}{5}, 0\right) = \frac{1}{5}(5\cos\theta, 5\sin\theta, 0) = (\cos\theta, \sin\theta, 0)$

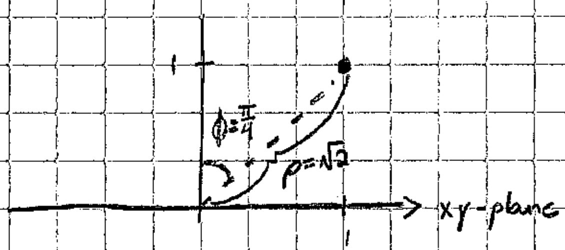
$0 \le z \le x + y + 10 = 5(\cos\theta + \sin\theta) + 10$

$$\Phi = \iint_S \vec{V}\cdot\hat{N}\,dS = \int_0^{2\pi}\int_0^{5(\cos\theta+\sin\theta)+10} (2\cos\theta + 3\sin\theta)\,5\,dz\,d\theta$$

$$= 5\int_0^{2\pi} (2\cos\theta + 3\sin\theta)(5\cos\theta + 5\sin\theta + 10)\,d\theta = 125\pi$$

12 In the book.

13 1.



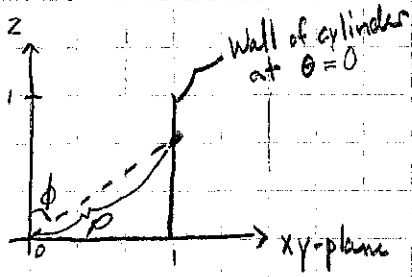
Since $0 \le \theta \le 2\pi$, this describes a circle of radius 1 about the z-axis at $z = 1$.

2. $\tan\phi = \sec\theta$

$\Rightarrow \cos\theta\sin\phi = \cos\phi$

$\Rightarrow \rho\cos\theta\sin\phi = \rho\cos\phi$

$\Rightarrow x = z$ This is a plane.

14 1. 

$$\sin\phi = \frac{1}{\rho}$$

$$\Longrightarrow \rho = \csc\phi \qquad \frac{\pi}{4} \le \phi \le \frac{\pi}{2} \qquad 0 \le \theta \le 2\pi$$

15 2. $\vec{V}(x,y,z) = (2,4,1)$ $\quad \hat{N} = (\cos\theta\sin\phi,\ \sin\theta\sin\phi,\ \cos\phi)$ $\quad dS = \rho^2 \sin\phi\, d\phi\, d\theta = 25\sin\phi\, d\phi\, d\theta$

$$\Phi = \iint_S \vec{V}\cdot\hat{N}\, dS = \int_0^{2\pi}\int_0^{\pi/2} (2\cos\theta\sin\phi + 4\sin\theta\sin\phi + \cos\phi)\, 25\sin\phi\, d\phi\, d\theta$$

$$= 25\pi$$

16 a) From symmetry, $\Phi = v_d$ (Area of hemisphere of radius R) where v_d is the velocity at a distance d from origin

$$\Longrightarrow v_0(2\pi R^2) = v_d(2\pi d^2)$$

$$\Longrightarrow v_d = \frac{v_0 R^2}{d^2}$$

See book solution to exercise 12.

b) $\Phi = V(2\pi R_1^2) = 2\pi V R_1^2$

17 $\rho^2 = x^2 + y^2 + z^2$

$$\iiint_D \rho^2\, dV = \int_0^{\pi/2}\int_0^{2\pi}\int_0^4 \rho^4 \sin\phi\, d\rho\, d\theta\, d\phi = \frac{2048\pi}{5}$$

18 $$Q = \int_0^{\pi}\int_0^{2\pi}\int_0^1 \left(\frac{10}{\pi}\rho\right)\rho^2 \sin\phi \, d\rho \, d\theta \, d\phi = 5$$

19 1. $$dS = \left|\left(1, 0, \frac{\partial z}{\partial x}\right) \times \left(0, 1, \frac{\partial z}{\partial y}\right)\right| dx\,dy$$

$$= \left|(1, 0, -2x) \times (0, 1, -2y)\right| dx\,dy = |(2x, -2y, 1)|\, dx\,dy$$

$$= (4x^2 + 4y^2 + 1)\, dx\,dy \underset{\substack{\uparrow \\ \text{convert to} \\ \text{polar}}}{=} (4r^2+1)\, r\,dr\,d\theta$$

$$\text{Surface Area} = \int_0^{2\pi}\int_0^1 (4r^2+1)\, r\,dr\,d\theta = 3\pi$$

21 1. $\nabla \cdot \vec{V} = \frac{\partial V_1}{\partial x} + \frac{\partial V_2}{\partial y} + \frac{\partial V_3}{\partial z} = 0 + 0 + 0 = 0$ This could represent the flow of an incomp. fluid w/ no source or sink

2. $\nabla \cdot \vec{V} = 1 + 1 + 1 = 3 \neq 0$ This could not.

22 $$\Phi = \iint_S \vec{V} \cdot d\vec{S} = \iiint_D \nabla \cdot \vec{V}\, dV$$ Gauss' Thm.

$\nabla \cdot \vec{V} = 1 + 1 = 2$ D is the sphere of radius 1

$$\Phi = \int_0^{\pi}\int_0^{2\pi}\int_0^1 2\rho^2 \sin\phi\, d\rho\, d\theta\, d\phi = \frac{8\pi}{3}$$

23 $\nabla \cdot \vec{V} = z + z = 2z$ D is the sphere of radius 4

$$\Phi = \int_0^{\pi}\int_0^{2\pi}\int_0^4 2z\rho^2 \sin\phi\, d\rho\, d\theta\, d\phi = \int_0^{\pi}\int_0^{2\pi}\int_0^4 2\rho^3 \cos\phi \sin\phi\, d\rho\, d\theta\, d\phi$$

$$= 0$$

24 $\Phi_{\substack{\text{whole}\\\text{cube}}} = \iint_S \vec{V} \cdot d\vec{S} = \iiint_D \nabla \cdot \vec{V} \, dV$

$\nabla \cdot \vec{V} = 3$

$\Phi_{\substack{\text{whole}\\\text{cube}}} = \int_{-2}^{2}\int_{-2}^{2}\int_{-2}^{2} 3 \, dx\,dy\,dz = 192$

$\Phi_{\text{face}} = \frac{1}{6} \Phi_{\substack{\text{whole}\\\text{cube}}} = 32$ Since the vector field is centrally symmetric.