Math 14
Winter 2009

A Few Notes on Writing Proofs

Here is a proof at the level of explanation you should be aiming for in the special homework. It is very relevant to your first special homework assignment.

Proposition: Suppose that $f : \mathbb{R} \to \mathbb{R}$. Then we can write f in the form

$$f(x) = cx$$

for some constant c, if and only if f satisfies the following two properties:

A. For all x and y, we have $f(x + y) = f(x) + f(y)$.

B. For all x and b we have $f(bx) = bf(x)$.

Proof:

First, assume that

$$f(x) = cx$$

for some constant c. We need to show that f satisfies (A) and (B).

To show (A), let x and y be any numbers. Then

$$f(x + y) = c(x + y) \text{ (by the definition of } f\text{)}$$

$$= cx + cy \text{ (by the distributive law)}$$

$$= f(x) + f(y) \text{ (by the definition of } f \text{ again),}$$

which is what we needed to show.

To show (B), let x be any number and b any constant. Then

$$f(bx) = c(bx) \text{ (by the definition of } f\text{)}$$

$$= (cb)x = (bc)x = b(cx) \text{ (by the associative and commutative laws)}$$

$$= bf(x) \text{ (by the definition of } f \text{ again)},$$

which is what we needed to show.

For the other direction, suppose that f satisfies (A) and (B). Let $c = f(1)$. We will show we can write $f(x) = cx$.

For any x we have

$$f(x) = f(x \cdot 1) = xf(1) \text{ (by property (B) where } b = x),$$

$$= f(1)x = cx.$$

This is what we needed to show.

Note: In your proof, you may need to use the following fact about matrix multiplication: If A and B are any matrices of the right shape for the product AB to be defined, and c is any scalar, then

$$A(cB) = c(AB) = (cA)B.$$

You can call this "the rule relating matrix multiplication to multiplication by a scalar."

Notice that a column vector is a special kind of matrix, so if B is a column vector, this rule still applies.

Another Note: Normally, you don't need to explain obvious steps like $(cb)x = c(bx)$. I included an explanation here because matrix multiplication is new to us, so the fact that matrix multiplication is associative does not yet count as obvious, and if you need to use the associative law for matrix multiplication (or some similar rule) in your proof, you should make a note of it.

General Comments:

The first thing to keep in mind is that mathematical writing is *writing*. You should write in complete English sentences, and you should explain things clearly so that your reader can understand what you are saying. Of course you may want to incorporate formulas, pictures, and diagrams into your writing. Formulas should be part of the text. A string of formulas, with no explanation of how you got from one to the next, is not a proof.

The second thing to keep in mind is that mathematical writing has its own special standards. Above all, good mathematical writing is clear to the point of transparency. You should be able to see right through the language to the ideas. Usually, simple sentence structure and repetitive word choice are good. This is not the place for fancy, poetic language—the poetry should be in the beauty and elegance of your mathematics.

Here are a few standard techniques that can help you organize proofs:

To prove a statement of the form “A is true if and only if B is true,” first prove, “If A is true then B is true,” and then prove, “If B is true then A is true.” Example:

> Claim: F is given by matrix multiplication if and only if F satisfies properties (1) and (2).
>
> Proof: First we show that if F is given by matrix multiplication, then F satisfies properties (1) and (2)...
>
> Now we show that if F satisfies properties (1) and (2), then F is given by matrix multiplication...

To prove a statement of the form “If A is true then B is true,” assume A is true and prove B is true. Example:

> We now show that if F is given by matrix multiplication, then F satisfies properties (1) and (2). Assume that F is given by matrix multiplication. In other words, we can write...

Alternatively, to prove a statement of the form “If A is true then B is true,” instead prove the *contrapositive*, “If B is false, then A is false.”

To prove a statement of the form “Either A is true or B is true,” you can prove “If A is false, then B is true.” Or, you can prove “It is not the case that A and B are both false.”

To prove a statement of the form "It is not the case that A is true," assume that A is true and prove a contradiction (a patently untrue statement, such as $0 = 1$). This is called *proof by contradiction*.

To prove a statement of the form "There is a number [or function, or...] for which A is true," find a number for which A is true, and prove A is true for that number. Example:

> We must show that we can write $F(\vec{v}) = A\vec{v}$ for some matrix A. Let A be the matrix given by
>
> $$A = \cdots$$
>
> and show that for every vector $\vec{v}$, we have $F(\vec{v}) = A\vec{v}$...

To prove a statement of the form "A is true of every number" let x (or the symbol of your choice) be a name for an arbitrary number, and prove A is true of x. Example:

> Claim: Every positive integer can be written as the sum of four squares.
>
> Proof: Let x be a positive integer...

Another technique that is sometimes useful is division into cases. Of course, you must make sure your cases cover all the possibilities. Example:

> Case 1: If x is odd, then ...
>
> Case 2: If x is even, then...

Finally, you can of course use theorems that have already been established. Examples:

> According to the Fundamental Theorem of Calculus...
>
> Theorem 5 on page 122 of the textbook says that...