Math 14
Winter 2009
Monday, February 9

Special Homework V

Instead of a new special homework, the February 9 assignment is to carefully rewrite Special Homework III. Here are some comments:

1. Your proof should be self-contained. Don't think of this as a homework problem, think of it as mini-paper. You wouldn't start a history term paper with, "Because of what we learned from Wednesday's lecture," so don't start this paper with, "Under the given assumptions."

   There is a copy of the original assignment after these comments. The assumptions on $\vec{r}$ in the first paragraph, and the definitions of t_i and Δt, all must be given before the statement of Proposition 1 makes sense. The further assumption (**) must be stated (and you must explain that (**) is known to be true under your assumptions, and so you will assume (**) also) before you can use (**) in a proof. The definitions of $R(n)$ and $S(n)$ must be given before the statement of Proposition 2 makes sense.

2. The statement of Proposition 1 begins "for every $\epsilon > 0$, there is an N such that for all $n > N$ ...." Thus, your proof should have the following structure: Let $\epsilon > 0$.... Let $N = \ldots$, and let $n > N$. Then show that n has the property you want.

3. If you are going to use the hint, explain to readers what you are doing before you start. Stating Proposition 1, and then jumping into "$|\vec{b}-\vec{a}| = \cdots$" makes readers wonder what $\vec{b}$ and $\vec{a}$ are; they feel as if they've skipped a page. (I thought John and Mary were alone in the room. Where did that encyclopedia salesman come from?) One way to handle this is to say, "First we will prove the following lemma." Another way, which one student used, is first to define $\vec{b}$ and $\vec{a}$ to be the vectors you are interested in. Even in this case, it makes sense to tell the reader, "First, we will show that ...."

4. Recall that (**) begins "for every $\epsilon > 0$ there is some $\delta > 0$ such that.... So the way you use (**) is to start with some $\epsilon > 0$, and get a $\delta > 0$. Possible language is something like, "Given $\epsilon > 0$, let $\delta > 0$ be as in (**). Then we define N by $N = \ldots$"

5. Proposition 2, with its proof using Proposition 1, gives us the most complicated logical structure. (I've simplified it a little by using ϵ and N in Proposition 1, and ϵ^* and N^* in Proposition 2. This is not usual; if I were writing for more experienced readers of mathematical proofs, I would use ϵ and N in both statements.) Proposition 1 says, "for every ϵ ... there is an N ...," and Proposition 2 says, "for every ϵ^* ... there is an N^* ...."

   The logical structure of your proof should reflect this: Start with an arbitrary ϵ^*. From it define some ϵ. Apply Proposition 1 to ϵ to get N. From this define N^*. Use what Proposition 1 tells you about N and ϵ to show that N^* works for ϵ^*.

6. Here are internet addresses for two papers about the writing of mathematics.

   The first is aimed at beginning calculus students writing very short papers for a calculus course. These papers are different from our writing assignments; in them, students generally adopt the role of mathematical consultant to an imaginary person with a problem that can be solved by the application of some calculus technique. However, many of the writing tips are relevant to anyone.
   http://edisk.fandm.edu/annalisa.crannell/writing_in_math/guide.html

   The second is aimed at mathematics majors writing a longer paper in the style of a mathematical journal. Again, this is not what we are doing, but a lot of the information on usage applies to all mathematical writing.
   http://www.mit.edu/
   afs/athena.mit.edu/course/other/mathp2/www/piil.html

Special Homework III Assignment

Suppose γ is a smooth curve parametrized by the function $\vec{r}(t)$ for $a \leq t \leq b$, and $\dfrac{d\vec{r}}{dt}$ is continuous on an open interval containing the closed interval $[a, b]$.

We said that if we take a small enough time interval Δt, the portion of the curve γ between $\vec{r}(t)$ and $\vec{r}(t + \Delta)$ should be almost straight. Therefore, we said, we should be able to approximate the arc length of γ in the following way: For some large n, divide the time interval $[a.b]$ into n-many equal subintervals of length $\Delta t = \dfrac{b-a}{n}$, with the endpoints of the subdivision being $t_0 = a$, $t_1 = a + \Delta t$, $t_2 = a + 2\Delta t$, $\ldots$, $t_n = a + n\Delta t = b$. Then we approximate the arc length of the portion of the curve between $\vec{r}(t_{i-1})$ and $\vec{r}(t_i)$ by the straight line distance between those two points,

$$(arclength)_i \approx |\vec{r}(t_i) - \vec{r}(t_{i-1})| = |\vec{r}(t_{i-1} + \Delta t) - \vec{r}(t_{i-1})|,$$

and approximate the entire arc length of γ by

$$S(n) = \sum_{i=1}^{n} |\vec{r}(t_i) - \vec{r}(t_{i-1})|.$$

Finally, we said the arc length of γ should be

$$\lim_{n\to\infty} S(n).$$

We further said that, if $\dfrac{d\vec{r}}{dt}$ represents velocity, and therefore $\left|\dfrac{d\vec{r}}{dt}\right|$ represents speed, then over a small interval from t to $t + \Delta t$, the direction of travel and the speed should not change very much, and so the distance traveled should be more or less the speed at time t times the elapsed time Δt, or $\left|\dfrac{d\vec{r}}{dt}(t)\right| \Delta t$. Therefore, we said that, if we divide the time interval $[a.b]$ into n-many equal subintervals as before, we should have, for each i,

$$|\vec{r}(t_i) - \vec{r}(t_{i-1})| = |\vec{r}(t_{i-1} + \Delta t) - \vec{r}(t_{i-1})| \approx (arclength)_i \approx \left|\frac{d\vec{r}}{dt}(t_{i-1})\right| \Delta t,$$

and we then concluded that we should have

$$R(n) = \sum_{i=1}^{n} \left| \frac{d\vec{r}}{dt}(t_{i-1}) \right| \Delta t \approx \sum_{i=1}^{n} |\vec{r}(t_i) - \vec{r}(t_{i-1})| = S(n),$$

and finally that

$$\lim_{n\to\infty} S(n) = \lim_{n\to\infty} R(n) = \int_a^b \left| \frac{d\vec{r}}{dt}(t) \right| dt.$$

There is a gap in this reasoning. It is not hard to see that, for a fixed t, as n gets larger and larger, the difference between $|\vec{r}(t+\Delta t) - \vec{r}(t)|$ and $\left| \frac{d\vec{r}}{dt}(t) \right| \Delta t$ gets smaller and smaller, because $\frac{d\vec{r}}{dt}(t)$ is just the limit as $\Delta t \to 0$ of $\frac{\vec{r}(t+\Delta t) - \vec{r}(t)}{\Delta t}$. However, even if the difference between the terms we are adding up gets smaller and smaller, the number of terms we are adding up to get $S(n)$ and $R(n)$ is getting larger and larger, so it is not obvious that the sums are getting closer and closer together.

To fill this gap we need one additional fact. Recall that by

$$\frac{d\vec{r}}{dt}(t) = \lim_{\Delta t \to 0} \frac{\vec{r}(t+\Delta t) - \vec{r}(t)}{\Delta t},$$

we mean that, for every t, for every $\epsilon > 0$, there is a $\delta > 0$ such that, whenever we have $0 < |\Delta t| < \delta$, then

$$\left| \frac{\vec{r}(t+\Delta t) - \vec{r}(t)}{\Delta t} - \frac{d\vec{r}}{dt}(t) \right| < \epsilon.$$

The value of ϵ may depend on the values of δ and t. However, given that $\frac{d\vec{r}}{dt}$ is continuous on an open interval containing the closed interval $[a, b]$, we can assume that we can choose ϵ in such a way that its value depends only on the value of δ and NOT on the value of t:

> (**) For every $\delta > 0$, there is an $\epsilon > 0$ such that, for every $t \in [a, b]$, whenever we have $0 < |\Delta t| < \delta$, then
>
> $$\left| \frac{\vec{r}(t+\Delta t) - \vec{r}(t)}{\Delta t} - \frac{d\vec{r}}{dt}(t) \right| < \epsilon.$$

Assignment: Prove the following propositions:

Proposition 1: Under the given assumptions, including (**), for every $\epsilon > 0$, there is an N large enough so that, for every $n > N$ and every i between 1 and n, we have

$$\left| |\vec{r}(t_{i-1} + \Delta t) - \vec{r}(t_{i-1})| - \left| \frac{d\vec{r}}{dt}(t_{i-1}) \right| \Delta t \right| < \epsilon \Delta t.$$

Hint for Proposition 1: First argue that, in general, if $|\vec{b} - \vec{a}| < \epsilon$ then $\big| |\vec{b}| - |\vec{a}| \big| < \epsilon$.

Proposition 2: Under the given assumptions, for every $\epsilon^* > 0$ there is an N^* large enough so that, for every $n > N^*$,

$$|S(n) - R(n)| < \epsilon^*.$$

Hint for Proposition 2: Apply Proposition 1 with $\epsilon = \dfrac{\epsilon^*}{b - a}$.