Math 14
Winter 2009
Monday, January 5

Special Homework I

We defined a function $F : \mathbb{R}^m \to \mathbb{R}^n$ to be *linear* if there is an $n \times m$ matrix A such that

$$F(\vec{v}) = A\vec{v}.$$

However, this is not the official definition of linear. Formally, a function F is linear if it satisfies the following two properties (for all vectors $\vec{v}$ and $\vec{w}$ and all scalars c):

A. $F(\vec{v} + \vec{w}) = F(\vec{v}) + F(\vec{w})$.

B. $F(c\vec{v}) = cF(\vec{v})$.

Assignment: Prove that a function $F : \mathbb{R}^2 \to \mathbb{R}^3$ satisfies properties (A) and (B) if and only if there is a 3×2 matrix A such that

$$F(\vec{v}) = A\vec{v}.$$

(We write both $\vec{v}$ and $F(\vec{v})$ as column vectors here.)

Hint: To show that if F satisfies (A) and (B) then F can be written $F(\vec{v}) = A\vec{v}$ for some matrix A, use the fact that we can write a vector $\vec{v}$ in the domain of F as

$$\vec{v} = \begin{pmatrix} v_1 \\ v_2 \end{pmatrix} = v_1 \begin{pmatrix} 1 \\ 0 \end{pmatrix} + v_2 \begin{pmatrix} 0 \\ 1 \end{pmatrix} = v_1\hat{\imath} + v_2\hat{\jmath},$$

and then write $F(\vec{v})$ in terms of $F(\hat{\imath})$ and $F(\hat{\jmath})$.

Note 1: If you have trouble with this proof, feel free to ask for help from fellow students, the instructor, or the TA. You must write up this proof yourself, but you can get all the help you want in figuring it out.

Note 2: You should be able to see how to generalize this proof from $F : \mathbb{R}^2 \to \mathbb{R}^3$ to $F : \mathbb{R}^m \to \mathbb{R}^n$.

Note 3: The definition "F is linear if F satisfies properties (A) and (B)" is preferable to our original definition, "F is linear if F is given by matrix multiplication," because it applies in a wider range of contexts. For example, let V be the collection of all continuous functions from the closed unit interval $[0, 1]$ to $\mathbb{R}$. Then the function F with domain V and range $\mathbb{R}$ that is defined by

$$F(f) = \int_0^1 f(x)\, dx$$

is linear because it satisfies properties (A) and (B):

$$\int_0^1 (f+g)(x)\, dx = \int_0^1 f(x)\, dx + \int_0^1 g(x)\, dx;$$

$$\int_0^1 cf(x)\, dx = c \int_0^1 f(x)\, dx.$$

Loosely speaking, a *vector space* is a collection of things that can be added together and multiplied by scalars. (This addition and multiplication have to satisfy certain rules, the vector space axioms.) The spaces $\mathbb{R}^n$ are vector spaces. So are the collection of continuous functions from $[0, 1]$ to $\mathbb{R}$, the collection of 3×2 matrices with entries from $\mathbb{R}$, and the collection of polynomials with rational coefficients. Whenever V and W are vector spaces, we define a function from V to W to be linear if it satisfies properties (A) and (B).