Math 14
Winter 2009
Sample Exam Proofs

Here are a couple of examples of proofs that might appear on exams, and that use only knowledge you should have from Math 8, together with acceptable answers.

Problem 1:

a. Show that if $\vec{v}$ is any vector function of t and $|\vec{v}|$ is constant, then $\vec{v}$ is normal (orthogonal, or perpendicular) to $\dfrac{d\vec{v}}{dt}$, provided they are both nonzero.

   Hint: Express $|\vec{v}|$ using the dot product, and remember that we have a "dot product rule" for differentiation.

b. Use the result of part (a) to show that if an object travels with constant speed, then its acceleration is normal to its direction of motion.

   This agrees with our physical intuition. Acceleration in the direction of motion should correspond to changing speed, and acceleration normal to the direction of motion should correspond to changing direction.

Problem 2: Use vectors to show that a parallelogram is a rectangle if and only if its two diagonals have the same length.

Solution to Problem 1:

a. Since $|\vec{v}|$ is constant, so is $|\vec{v}|^2$, and so its derivative is zero.

$$\frac{d}{dt}|\vec{v}|^2 = 0$$

$$\frac{d}{dt}(\vec{v}\cdot\vec{v}) = 0$$

Now we use the dot product rule.

$$\vec{v}\cdot\left(\frac{d\vec{v}}{dt}\right) + \left(\frac{d\vec{v}}{dt}\right)\cdot\vec{v} = 0$$

$$2\vec{v}\cdot\left(\frac{d\vec{v}}{dt}\right) = 0$$

$$\vec{v}\cdot\left(\frac{d\vec{v}}{dt}\right) = 0$$

Since the dot product is zero, $\vec{v} \perp \frac{d\vec{v}}{dt}$ (provided they are both nonzero).

b. If $\vec{v}$ is the velocity of our moving object, then the speed is $|\vec{v}|$, so since the speed is constant, by part (a) we have $\vec{v} \perp \frac{d\vec{v}}{dt}$. Acceleration is the derivative of velocity $\frac{d\vec{v}}{dt}$, so acceleration is normal to velocity. The direction of motion is the direction of the velocity vector, so acceleration is normal to the direction of motion.

Solution to Problem 2:

If $\vec{u}$ and $\vec{v}$ are the two edges of a parallelogram, then the diagonals of the parallelogram are $\vec{u}+\vec{v}$ and $\vec{u}-\vec{v}$. (See picture.) The parallelogram is a rectangle just in case its edges are perpendicular. So we must show that

$$|\vec{u}+\vec{v}| = |\vec{u}-\vec{v}| \iff \vec{u} \perp \vec{v}.$$

To show this:

$$\begin{aligned} |\vec{u}+\vec{v}| = |\vec{u}-\vec{v}|. &\iff |\vec{u}+\vec{v}|^2 = |\vec{u}-\vec{v}|^2 \\ &\iff (\vec{u}+\vec{v})\cdot(\vec{u}+\vec{v}) = (\vec{u}-\vec{v})\cdot(\vec{u}-\vec{v}) \\ &\iff \vec{u}\cdot\vec{u} + 2(\vec{u}\cdot\vec{v}) + \vec{v}\cdot\vec{v} = \vec{u}\cdot\vec{u} - 2(\vec{u}\cdot\vec{v}) + \vec{v}\cdot\vec{v} \\ &\iff 2(\vec{u}\cdot\vec{v}) = -2(\vec{u}\cdot\vec{v}) \\ &\iff \vec{u}\cdot\vec{v} = 0 \\ &\iff \vec{u} \perp \vec{v}, \end{aligned}$$

which was what we needed to prove.