Math 14
Winter 2009
Possible Exam Questions

1. Complete these definitions:

   (a) If F is a function from $\mathbb{R}^m$ to $\mathbb{R}^n$, then
   $$\lim_{\vec{x}\to\vec{x}_0} F(\vec{x}) = \vec{a}$$
   means:

   (b) A function F from $\mathbb{R}^m$ to $\mathbb{R}^n$ is *linear* if:

   (c) A function F from $\mathbb{R}^m$ to $\mathbb{R}^n$ is *differentiable* at the point $\vec{x}_0$ if:

   (d) The graphs of the functions $F(\vec{x})$ and $A(\vec{x})$ from $\mathbb{R}^m$ to $\mathbb{R}^n$ are *tangent* at the point $\vec{x}_0 \in \mathbb{R}^m$ if:

   (e) The *open ball* of radius r around the point $\vec{x}_0$ in $\mathbb{R}^m$ is:

2. State the Chain Rule.

3. A rectangular cardboard box with an open top is to hold 16 cubic inches. Give the dimensions the box should have in order to use the minimal amount of cardboard.

4. You may use this formula for the inverse of a 2×2 matrix: If
   $$A = \begin{pmatrix} a & b \\ c & d \end{pmatrix}$$
   and $det(A) \neq 0$ then
   $$A^{-1} = \frac{1}{det(A)} \begin{pmatrix} d & -b \\ -c & a \end{pmatrix}$$

   (a) Near which points (u, v) is the function $F(u, v) = (u^2 + v^2, uv)$ invertible?

   (b) Find the derivative $(F^{-1})'(5, 2)$ where F^{-1} is the inverse function of F near the point $(u, v, x, y) = (1, 2, 5, 2)$.

5. Motion along a helix in $\mathbb{R}^3$ is given by the position function $\vec{r}(t) = \langle \cos t, \sin t, t \rangle$.

   (a) Find the velocity and acceleration of the moving object at time t.

   (b) Express the acceleration of the moving object as the sum of two components, one parallel to the direction of motion and the other orthogonal to the direction of motion.

   (c) Find the curvature of the helix at the point $\vec{r}(t)$.

6. A curve C in $\mathbb{R}^2$ is given by the equation

$$g(u, v) = 0,$$

   where $u = x^2 + y^2$ and $v = xy$. If the point $(x, y) = (1, 2)$ is on C, and if we have

$$\frac{\partial g}{\partial u} = u + v \qquad \frac{\partial g}{\partial v} = u - v$$

   find the tangent line to C at the point $(x, y) = (1, 2)$.

7. Suppose $S \subseteq \mathbb{R}^3$ is the graph of a differentiable function $f : \mathbb{R}^2 \to \mathbb{R}$, and (a, b, c) is a point on S.

   (a) Under what conditions does S determine x as a function $h(y, z)$ of y and z near (a, b, c)?

   (b) Under these conditions, what is the value of $\dfrac{\partial h}{\partial y}(b, c)$?

   (Your answers should be phrased in terms of f and its partials.)

8. (a) Complete the definition: A function F from $\mathbb{R}^m$ to $\mathbb{R}^n$ is *linear* if...

   (b) Prove that the composition of two linear functions is again linear.

9. The directional derivative of a function $f : \mathbb{R}^n \to \mathbb{R}$ at the point $\vec{x} \in \mathbb{R}^n$ in the direction of the unit vector $\vec{u}$ is defined to be

$$D_{\vec{u}} f(\vec{x}) = \frac{d}{dt} f(\vec{x} + t\vec{u})\Big|_{t=0}.$$

   Use the Chain Rule to show that, assuming f is differentiable on some open neighborhood of $\vec{x}$, we can write

$$D_{\vec{u}} f(\vec{x}) = \nabla f(\vec{x}) \cdot \vec{u}.$$

10. Use a tangent approximation to the function

$$(r, \theta) = F(x, y) = \left(\sqrt{x^2 + y^2},\ \tan^{-1}\left(\frac{y}{x}\right)\right)$$

to approximate the polar coordinates of the point $(x, y) = (1.02, .99)$.

11. Use the definition of limit to show that

$$\lim_{(x,y)\to(0,0)} \frac{x^2 - y^2}{x^2 + y^2}$$

does not exist.

12. Find the total charge on the top half of the unit circle if the linear charge density at point (x, y) is y^3.