Math 14
Winter 2009
Monday, January 12

An Example

These problems will require you to show some things about limits. You do NOT have to use epsilons and deltas. You may use any valid reasoning, and any theorems you know. If you use the polar coordinates trick, you might want to remember the double angle formula $sin(2\theta) = 2(\sin\theta)(\cos\theta)$.

Define a function $f : \mathbb{R}^2 \to \mathbb{R}$ by

$$f(x,y) = \begin{cases} \frac{2xy}{\sqrt{x^2+y^2}} & (x,y) \neq (0,0); \\ 0 & (x,y) = (0,0. \end{cases}$$

(1.) Show that f is continuous at $(0,0)$.

(2.) Find the partial derivatives of f. At the origin, you will have to use the definition of partial derivative,

$$\frac{\partial f}{\partial x}(0,0) = \lim_{x \to 0} \frac{f(x,0) - f(0,0)}{x - 0};$$

$$\frac{\partial f}{\partial y}(0,0) = \lim_{y \to 0} \frac{f(0,y) - f(0,0)}{y - 0}.$$

(3.) Show the partial derivatives of f are not continuous at $(0, 0)$.

(4.) If the graph of f has a tangent plane at the origin, it must be the graph of the function

$$g(x, y) = \left(\frac{\partial f}{\partial x}(0,0)\right)(x-0) + \left(\frac{\partial f}{\partial y}(0,0)\right)(y-0) + f(0,0) = \underline{\qquad\qquad}.$$

Show the graphs of f and g are not tangent at the origin, by showing

$$\lim_{(x,y)\to(0,0)} \frac{f(x,y) - g(x,y)}{|(x,y)-(0,0)|} \neq 0.$$

(5.) Show that if

$$\vec{u} = \left(\frac{\sqrt{2}}{2}, \frac{\sqrt{2}}{2} \right),$$

then the directional derivative of f at $(0,0)$ in the direction of $\vec{u}$,

$$\lim_{t \to 0} \frac{f((0,0) + t\vec{u}) - f(0,0)}{t},$$

is undefined.

Does it make sense that we say f is not differentiable at the origin?

(6.) Sketch the intersection of the graph of f with the plane $x = y$. Your picture should look like a two-dimensional graph, where the points on the horizontal axis are $(x, y) = (1, 1)$, $(x, y) = (2, 2)$, and so forth.

Does it make sense that we say f is not differentiable at the origin?