Math 14
Winter 2009
Friday, January 9

Differentiability

Definition: If $F : \mathbb{R}^m \to \mathbb{R}^n$ and some open ball around $\vec{a}$ is contained in the domain of F (we sometimes phrase this as "F is defined at and near $\vec{a}$"), then we say F is *differentiable* at $\vec{a}$ if there is an affine (linear plus constant) function

$$T(\vec{x}) = A(\vec{x} - \vec{a}) + \vec{b}$$

whose graph is tangent to the graph of F at $\vec{x} = \vec{a}$.

Here A is an $n \times m$ matrix whose entries are constants, $\vec{x}$ and $\vec{a}$ are written as column vectors, and $\vec{b}$ is an n-entry column vector.

Example:

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Definition: The graph of T is tangent to the graph of F at $\vec{x} = \vec{a}$ if

$$T(\vec{a}) = F(\vec{a});$$

$$\lim_{\vec{x} \to \vec{a}} \frac{F(\vec{x}) - T(\vec{x})}{|\vec{x} - \vec{a}|} = \vec{0}.$$

Theorem: If $F : \mathbb{R}^m \to \mathbb{R}^n$ is differentiable at $\vec{a}$ with tangent approximation $T(\vec{x}) = A(\vec{x} - \vec{a}) + \vec{b}$, then $\vec{b} = F(\vec{a})$ and A is the matrix of partial derivatives of F at the point $\vec{a}$

$$F(\vec{x}) = \langle F_1(x_1, \ldots, x_m), \ldots, F_n(x_1, \ldots, x_m) \rangle\,;$$

$$A = \begin{pmatrix} \frac{\partial F_1}{\partial x_1} & \frac{\partial F_1}{\partial x_2} & \cdots & \frac{\partial F_1}{\partial x_m} \\ \frac{\partial F_2}{\partial x_1} & \frac{\partial F_2}{\partial x_2} & \cdots & \frac{\partial F_2}{\partial x_m} \\ \vdots & \vdots & \ddots & \vdots \\ \frac{\partial F_n}{\partial x_1} & \frac{\partial F_n}{\partial x_2} & \cdots & \frac{\partial F_n}{\partial x_m} \end{pmatrix}.$$

Note that A has one row for each coordinate function of F, and one column for (the partial derivative with respect to) each coordinate of the argument $\vec{x}$.

Definition: If $F : \mathbb{R}^m \to \mathbb{R}^n$ is differentiable at $\vec{a}$ with tangent approximation $T(\vec{x}) = A(\vec{x} - \vec{a}) + \vec{b}$, then we say

$$F'(\vec{a}) = A = \begin{pmatrix} \frac{\partial F_1}{\partial x_1} & \frac{\partial F_1}{\partial x_2} & \cdots & \frac{\partial F_1}{\partial x_m} \\ \frac{\partial F_2}{\partial x_1} & \frac{\partial F_2}{\partial x_2} & \cdots & \frac{\partial F_2}{\partial x_m} \\ \vdots & \vdots & \ddots & \vdots \\ \frac{\partial F_n}{\partial x_1} & \frac{\partial F_n}{\partial x_2} & \cdots & \frac{\partial F_n}{\partial x_m} \end{pmatrix}.$$

Warning: A function can have partial derivatives at a point and still not be differentiable there.