Math 14
Winter 2009
Friday, January 9

More About Limits

Something to do if you finish the quiz early: Fill in the missing parts of the following proofs.

Theorem: Suppose that $F : \mathbb{R}^m \to \mathbb{R}^n$ is written in terms of its component functions as

$$F(\vec{x}) = \langle F_1(\vec{x}),\, F_2(\vec{x}),\, \ldots,\, F_n(\vec{x}) \rangle\,,$$

that the domain of F is an open set A, that $\vec{a}$ is either in A or a boundary point of A, and that the domain of each F_i is A. Show that if

$$\lim_{\vec{x} \to \vec{a}} F(\vec{x}) = \vec{b} = \langle b_1,\, b_2,\, \ldots,\, b_n \rangle\,,$$

then for every i between 1 and n we have

$$\lim_{\vec{x} \to \vec{a}} F_i(\vec{x}) = b_i.$$

Proof: Let i be given. (That is, consider some particular i.) We need to show that for every $\epsilon > 0$, there is a $\delta > 0$ such that the following is true:

Because $\lim_{\vec{x}\to\vec{a}} F(\vec{x}) = \vec{b}$, we know that for our given ϵ there is some $\overline{\delta} > 0$ such that whenever $\vec{x}$ is in A and $\vec{x} \neq \vec{a}$, we have

$$|\vec{x} - \vec{a}| < \overline{\delta} \implies |F(\vec{x}) - \vec{b}| < \epsilon.$$

(We say $\overline{\delta}$ because the distance that works for F may or may not be the same δ we are looking for that works for F_i.)

If $|F(\vec{x}) - \vec{b}| < \epsilon$ then we know that

$$|F_i(\vec{x}) - b_i| < \underline{\hspace{3cm}}.$$

Therefore we can take

$$\delta = \underline{\hspace{3cm}}.$$

Theorem: Suppose that $F : \mathbb{R}^m \to \mathbb{R}^n$ is written in terms of its component functions as

$$F(\vec{x}) = \langle F_1(\vec{x}),\, F_2(\vec{x}),\, \ldots,\, F_n(\vec{x}) \rangle \,,$$

that the domain of F is an open set A, that $\vec{a}$ is either in A or a boundary point of A, and that the domain of each F_i is A. Show that if for every i between 1 and n

$$\lim_{\vec{x} \to \vec{a}} F_i(\vec{x}) = b_i,$$

then we have

$$\lim_{\vec{x} \to \vec{a}} F(\vec{x}) = \vec{b} = \langle b_1,\, b_2,\, \ldots,\, b_n \rangle \,.$$

Proof: Let $\epsilon > 0$. We need to show that there is a $\delta > 0$ such that whenever $\vec{x}$ is in A and $\vec{x} \neq \vec{a}$, we have

$$|\vec{x} - \vec{a}| < \delta \implies |F(\vec{x}) - \vec{b}| < \epsilon.$$

We know that

$$|F(\vec{x}) - \vec{b}| = \sqrt{\sum_{i=1}^{n} (F_i(\vec{x}) - b_i)^2} \leq \sum_{i=1}^{n} |F_i(\vec{x}) - b_i|,$$

so if we can make each $|F_i(\vec{x}) - b_i|$ less than $\dfrac{\epsilon}{n}$, we will have succeeded.

We know that for each i, there is a δ_i such that the following is true:

Therefore we can take

$$\delta = \underline{\hspace{3cm}}.$$

Theorem: Suppose that $F : \mathbb{R}^m \to \mathbb{R}^n$ is written in terms of the components of its argument as

$$F(\vec{x}) = F(\langle x_1,\, x_2,\, \ldots,\, x_m \rangle),$$

that the domain of F is an open set A, and that $\vec{a} = \langle a_1,\, a_2,\, \ldots a_m \rangle$ is either in A or a boundary point of A. Suppose also that if $\vec{a}$ is a boundary point of A, there is some open ball B around $\vec{a}$ such that every point of B except $\vec{a}$ is in A. Show that if

$$\lim_{\vec{x} \to \vec{a}} F(\vec{x}) = \vec{b},$$

then for every i between 1 and m we have

$$\lim_{x_i \to a_i} F(\langle a_1,\, a_2,\, \ldots,\, a_{i-1},\, x_i,\, a_{I+1},\, \ldots,\, a_m \rangle) = \vec{b}.$$

(That is, as $\vec{x} \to \vec{a}$ along the line given by $x_1 = a_1, x_2 = a_2, \ldots, x_{i-1} = a_{i-1}, x_{i+1} = a_{i+1}, \ldots x_m = a_m$, we have $F(\vec{x}) \to \vec{b}$.)

Proof:

Question for the very careful: Why did we assume that if $\vec{a}$ is not in the domain of F, there is an open ball B around $\vec{a}$ such that everything in B except $\vec{a}$ is in the domain of F?

Hint: What if the domain of F consists of all points on the xy-plane that are not on the x-axis and we know

$$\lim_{(x,y)\to(0,0)} F(x,y) = (2,3,2)?$$

Can we conclude that

$$\lim_{x\to 0} F(x,0) = (2,3,2)?$$

Why or why not?